A space-filling curve

In mathematics classes where the natural numbers are studied, several surprising results can be shown about the relative "sizes" of infinite sets, such as the fact the there is a one-to-one mapping between the natural numbers $\mathbb{N}$ and the even numbers, showing that they are both countably infinite, with the "same size". Similar logic also shows that there is a mapping from the natural numbers $\mathbb{N}$ onto the product space $\mathbb{N} \times \mathbb{N}$ of all pairs of natural numbers.

When dealing with real numbers, it should therefore be somewhat expected that there is a mapping from the unit interval $[0,1]$ onto the unit square $[0,1]^2$. However, what is surprising is that there continuous mappings of this form. These are frequently referred to as space-filling curves, or Peano curves, after the Italian mathematician Giuseppe Peano (1858–1932). Here, a specific space-filling curve due to Schoenberg [1] is described. As can been seen in the graphs below, this curve has a complex and overlapping structure, and it is possible to construct curves with a much more regular structure. However, the curve by Schoenberg has the advantage of being relatively straightforward to analyze.

To begin, a continuous function $f : \mathbb{R} \to [0,1]$ is introduced, which is even and is periodic with period 2, so that $f(x) = f(x+2)$ for all x. On the interval from $[0,1]$,

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1/3, \\ 3x - 1 & \text{if } 1/3 \le x < 2/3, \\ 1 & \text{if } 2/3 \le x < 1. \end{cases}$$

The function is plotted in Fig. 1. Define two sequences of functions according to

$$x_n(t) = \sum_{k=1}^{n} \frac{f(3^{2k-2})t)}{2^k}, \qquad y_n(t) = \sum_{k=1}^{n} \frac{f(3^{2k-1}t)}{2^k}.$$

Since each of the x_n is a finite sum of terms which are each scalings of f, they are continuous. Consider the kth term of $x_n(t)$:

$$\left| \frac{f(3^{2k-2}t)}{2^k} \right| \le 2^{-k}.$$

Since $\sum_{k=1}^{\infty} 2^{-k} = 1$, the Weierstraß M-test shows that $x_n(t)$ converges uniformly to a function $x(t)$. In addition, it has been shown that a uniformly convergent sequence of continuous functions has a continuous limit, so $x(t)$ is a continuous function. The same argument can be applied to show that $y_n(t)$ converges to a continuous function $y(t)$.

Now consider the mapping $g : [0,1] \to [0,1]^2$ specified by $g(t) = (x(t), y(t))$. To show that this a mapping onto $[0,1]^2$, consider any point $(x_0, y_0) \in [0,1]^2$. In the same way that any real number can be written as a decimal expansion, the values of x_0 and y_0 have binary expansions of the form

$$x_0 = \sum_{k=1}^{\infty} \frac{a_k}{2^k}, \qquad y_0 = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$$

where the $a_k, b_k \in \{0,1\}$. A new sequence (c_k) can be defined by interleaving these two sequences according to $c_{2k-1} = a_k$ and $c_{2k} = b_k$ for $k \in \mathbb{N}$. Consider the value

$$t_0 = 2\sum_{k=1}^{\infty} \frac{c_k}{3^k}.$$

Then

$$\begin{aligned}
x(t_0) &= \sum_{k=1}^{\infty} 2^{-k} f\left(3^{2k-2} 2\sum_{j=1}^{\infty} c_j 3^{-j}\right) \\
&= \sum_{k=1}^{\infty} 2^{-k} f\left(2\sum_{j=1}^{\infty} c_j 3^{2k-2-j}\right) \\
&= \sum_{k=1}^{\infty} 2^{-k} f\left(2\sum_{j=1}^{2k-2} c_j 3^{2k-2-j} + 2c_{2k-1}3^{-1} + 2\sum_{j=2k}^{\infty} c_j 3^{2k-2-j}\right).
\end{aligned}$$

Consider the argument of f: the first term is an even integer, and thus by the periodicity of f, this term plays no role. The third term is positive, and for any $N \in \mathbb{N}$ with $N \geq 2k$,

$$\begin{aligned}
2\sum_{j=2k}^{N} c_j 3^{2k-2-j} &= 2\sum_{j=0}^{N} c_j 3^{-2-j} \\
&\leq \frac{2}{9}\sum_{j=0}^{N} 3^{-j} \\
&\leq \frac{2}{9}\frac{1}{1-\frac{1}{3}} = \frac{1}{3}.
\end{aligned}$$

Thus if $c_{2k-1} = 0$, then second two terms are between 0 and $1/3$, and if $c_{2k-1} = 1$, then they are between $2/3$ and 1. By using the definition of f, it follows that

$$f\left(2\sum_{j=1}^{2k-2} c_j 3^{2k-2-j} + 2c_{2k-1}3^{-1} + 2\sum_{j=2k}^{\infty} c_j 3^{2k-2-j}\right) = c_{2k-1}$$

and hence

$$x(t_0) = \sum_{k=1}^{\infty} \frac{c_{2k-1}}{2^k} = \sum_{k=1}^{\infty} \frac{a_k}{2^k} = x_0.$$

The same approach can be applied to show that $y(t_0) = y_0$. Hence g maps onto $[0,1]^2$. Figures 2, 3, and 4 show plots of the first five partial sums $(x_k(t), y_k(t))$. Even for $k = 1, 2$ the curves become complex to follow. However, it can be seen the kth curve passes through all points in the grid of spacing 2^{-k}. Figure 5 shows the evolution of the individual functions $x_4(t)$ and $y_4(t)$.

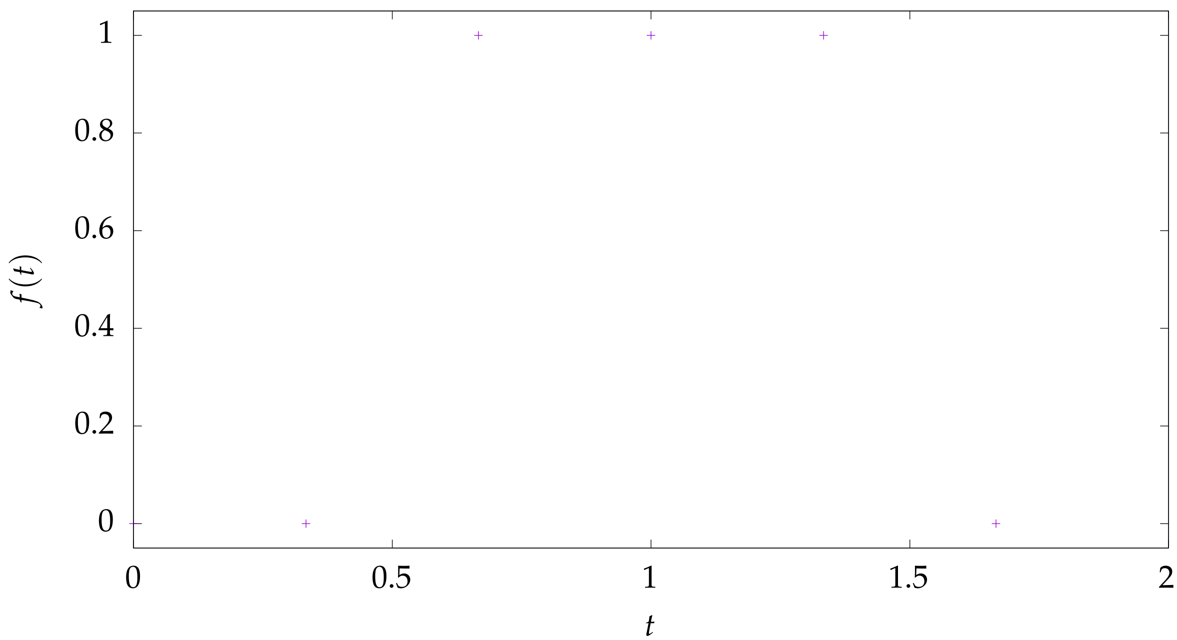

Figure 1: A graph of the function $f(t)$ on which the space-filling curve is based.

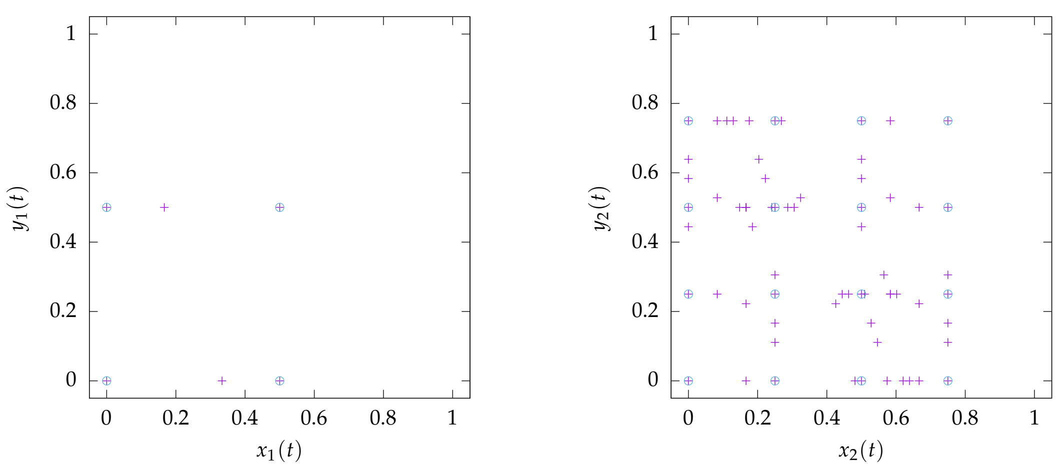

Figure 2: Graphs of the curves given by $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$. Superimposed on the two curves are grids with spacing $1/2$ and $1/4$, showing that the paths pass through all of these points.

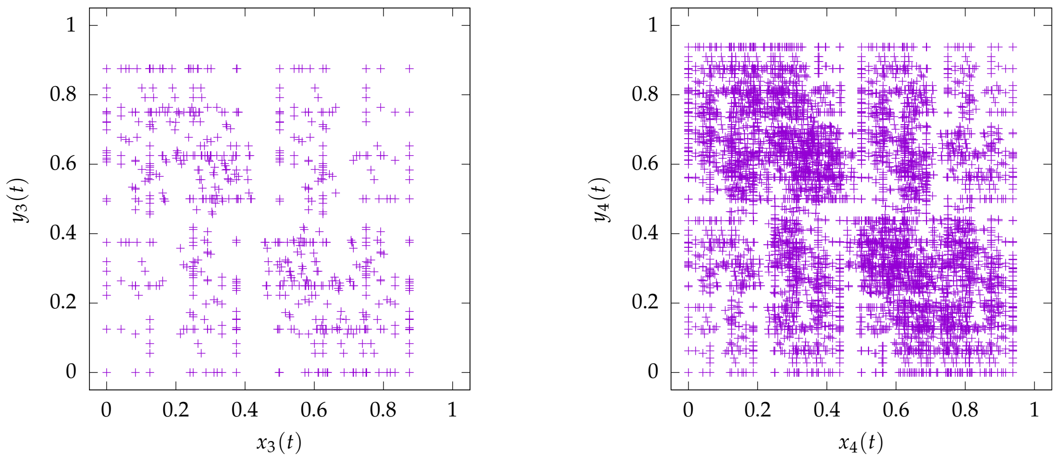

Figure 3: Graphs of the curves given by $(x_3(t), y_3(t))$ and $(x_4(t), y_4(t))$

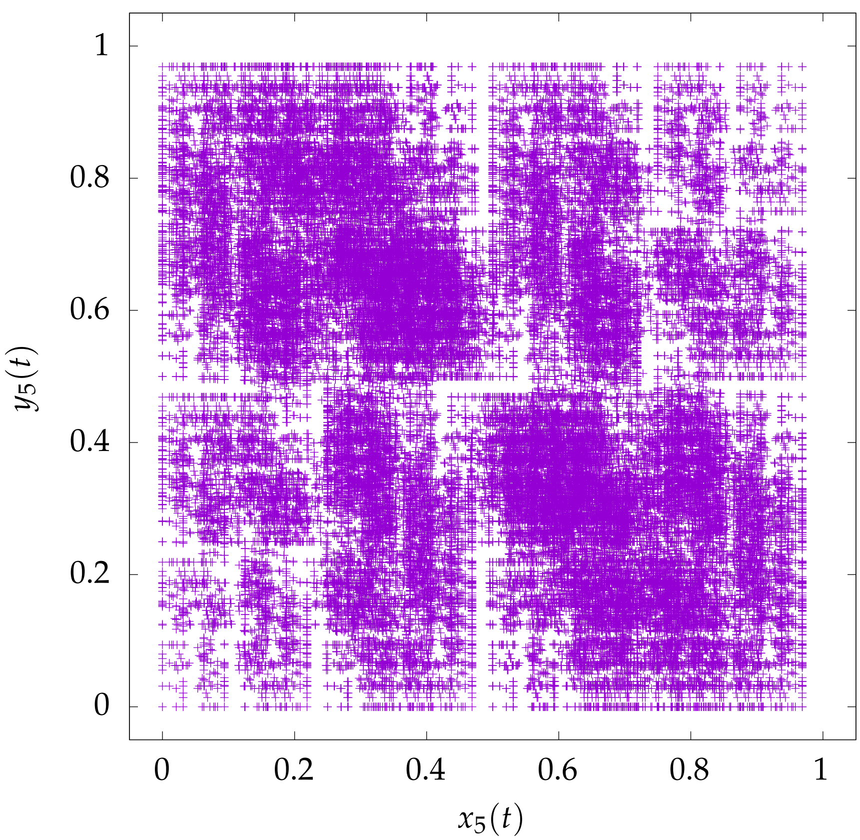

Figure 4: A graph of the curve $(x_5(t), y_5(t))$.

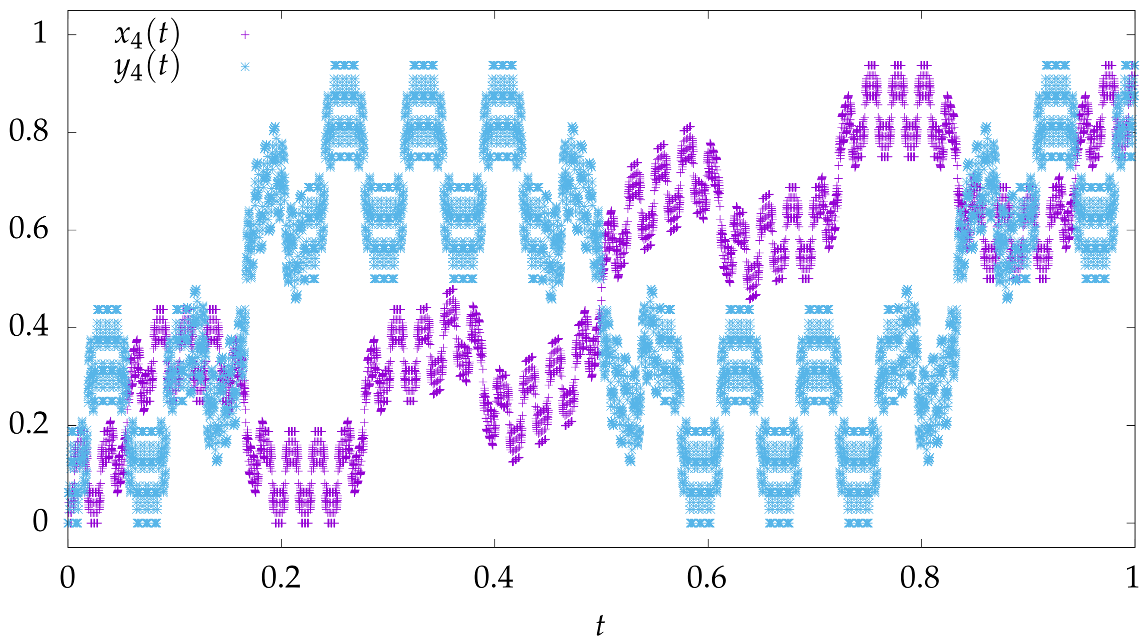

Figure 5: A graph of the functions $x_4(t)$ and $y_4(t)$.

References

[1] I. J. Schoenberg, *On the Peano curve of Lebesgue*, Bull. Amer. Math. Soc. **44** (1938) 519. doi:10.1090/S0002-9904-1938-06792-4.