Math 104: Midterm 2 sample solutions

1. To show that f is uniformly continuous, choose $\epsilon > 0$. Since $f_n \to f$ uniformly, there exists an N such that $n > N$ implies that
$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$
for all $x \in (a,b)$. Now consider f_{N+1}: since this is uniformly continuous, there exists a $\delta > 0$ such that if $x, y \in (a,b)$ and $|x-y| < \delta$, then
$$|f_{N+1}(x) - f_{N+1}(y)| < \frac{\epsilon}{3}.$$
Now, for any $x, y \in (a,b)$ with $|x-y| < \delta$,
$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(y)| + |f_{N+1}(y) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$
and hence f is uniformly continuous.

2. For three values 0, 1, and 2,
$$d_1(0,1) + d_1(1,2) = 1^4 + 1^4 = 2$$
but
$$d_1(0,2) = 2^4 = 16$$
and hence the triangle inequality is violated, so d_1 is not a metric.

   Since $d_2(0,0) = 1$, it does not satisfy the property that $d(x,x) = 0$ for all $x \in \mathbb{R}$, and hence d_2 is not a metric. Since $d_3(0,1) = 2$, and $d_3(1,0) = 1$ it is not symmetric, and hence it is not a metric.

3. First, consider the series
$$\sum_{n=0}^{\infty} \frac{x^n}{n^{\sqrt{n}}}$$
so that the coefficients are $a_n = n^{-\sqrt{n}}$. Then
$$\begin{aligned} \beta &= \limsup |a_n|^{1/n} \\ &= \limsup |n^{-\sqrt{n}}|^{1/n} \\ &= \limsup n^{-1/\sqrt{n}}. \end{aligned}$$
Since the power will always be negative, all of the terms in this sequence must be less than or equal to 1, so $\beta \leq 1$. Consider the subsequence of terms for $n = 2^k$. Then
$$n^{-1/\sqrt{n}} = 2^{-k/2^{k/2}}.$$

It is known that $k/2^{k/2} \to 0$ as $k \to \infty$. Hence $2^{-k/2^{-k}} \to 1$ as $k \to \infty$. Hence, since the subsequence tends to 1 as $n \to \infty$, then $\beta \geq 1$. Combining with the result above, $\beta = 1$, and the radius convergence is $R = 1$.

For the second series

$$\sum_{n=0}^{\infty} 4^n \, x^{2n+1}$$

then

$$a_n = \begin{cases} 2^{n-1} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Consider the subsequence of odd terms:

$$|a_{2k+1}|^{1/(2k+1)} = |2^{2k+1-1}|^{1/(2k+1)} = 2^{2k/(2k+1)}$$

which converges to 2 as $k \to \infty$. Hence

$$\beta = \limsup |a_n|^{1/n} = 2$$

and therefore the radius of convergence is $R = 1/2$.

For the third series,

$$\sum_{n=0}^{\infty} x^{n^2},$$

the coefficients are $a_n = 1$ if n is a square number and zero otherwise. Hence

$$|a_n|^{1/n} = \begin{cases} 1 & \text{if } n \text{ is a square number} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\beta = \limsup |a_n|^{1/n} = 1$$

since there are infinitely many terms which are 1 in the sequence. Hence the radius of convergence is $R = 1$.

4. If g is bounded on S, then there exists an $M > 0$ such that $|g(x)| < M$ for all $x \in S$. Now consider any $\epsilon > 0$. If f_n converges uniformly to f, then there exists an N such that $n > N$ implies that

$$|f_n(x) - f(x)| < \frac{\epsilon}{M}$$

for all $x \in S$. Now consider $g \cdot f_n$:

$$|g(x)f_n(x) - g(x)f(x)| = |g(x)| \cdot |f_n(x) - f(x)| < M\frac{\epsilon}{M} = \epsilon$$

and hence it uniformly converges to $g \cdot f$.

5. If f_n converges uniformly to f, then there exists an N such that $n > N$ implies that

$$|f_n(x) - f(x)| < 1$$

for all $x \in S$. By using the triangle inequality,

$$|f(x)| < |f_{N+1}(x)| + 1.$$

Since f_{N+1} is bounded, $|f_{N+1}(x)| < M$ for all x and for some $M \geq 0$, and thus $|f(x)| < M + 1$ for all x. Hence f is bounded.

6. Since a continuous function on a closed interval is bounded, then for each f_n, there exists M_n such that $|f_n(x)| < M_n$ for all $x \in [0, 1]$. Since f_n converges uniformly to f, there exists an $N \in \mathbb{N}$ such that for all $n > N$,

$$|f_n(x) - f(x)| \leq 1$$

Since the f_n converge uniformly and are continuous, the limit f is continuous also, and therefore bounded, so that $|f(x)| < M'$ for all $x \in [0, 1]$, for some $M' > 0$. By using the triangle inequality,

$$|f_n(x)| < |f(x)| + 1 < M' + 1$$

for all $n > N$. Now define

$$L = \max\{M_1, M_2, \ldots, M_N, M' + 1\}$$

Consider any $x \in [0, 1]$ and any $n \in \mathbb{N}$. If $n \leq N$, then $|f_n(x)| < M_n \leq L$. If $n > N$, then $|f_n(x)| < M' + 1 \leq L$. Thus L is an upper bound for the set $A = \{|f_n(x)| : n \in \mathbb{N},\ x \in [0, 1]\}$. Hence $0 \leq \sup A \leq L$, and thus $\sup A$ must be finite.