Math 104: final information

- The final exam will take place on Wednesday May 11th, from 7pm–10pm.
- The exam will cover all parts of the course with equal weighting. It will cover Chapters 1–5, 7–15, 17–21, 23–34, 36, 37 of Ross.
- The final will consist of eleven questions, which will be of a similar style to those on the midterms. This will give you slightly longer per question than on the midterms.
- The exam will be closed book – no textbooks, notebooks, or calculators allowed. As on the midterms, you will be expected to be familiar with the basic defintions, and know the key results, but it will not be necessary to remember the proof of every theorem by heart.
- As on the midterms, the basic results listed below can be used without proof.
  - ◇ $\sum n^{-p}$ diverges if and only if $p > 1$,
  - ◇ the triangle inequality: $|a| + |b| \geq |a + b|$ for all $a, b \in \mathbb{R}$,
  - ◇ $\lim_{n\to\infty} a^n = 0$ for $|a| < 1$,
  - ◇ $|b| < a$ if and only if $-a < b < a$.

Sample final questions

1. Consider the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^3}, \qquad \sum_{n=1}^{\infty} \frac{x^{3n}}{2n}, \qquad \sum_{n=0}^{\infty} x^{2n!}.$$

   For each power series, determine its radius of convergence R. By considering the series at $x = \pm R$, determine the exact interval of convergence.

2. (a) Prove by using the definition of convergence only, without using limit theorems, that if (s_n) is a sequence converging to s, then $\lim_{n\to\infty} s_n^2 = s^2$.

   (b) Prove by using the definition of continuity, or by using the ϵ-δ property, that $f(x) = x^2$ is a continuous function on $\mathbb{R}$.

3. Let f be a twice differentiable function defined on the closed interval $[0, 1]$. Suppose $r, s, t \in [0, 1]$ are defined so that $r < s < t$ and $f(r) = f(s) = f(t) = 0$. Prove that there exists an $x \in (0, 1)$ such that $f''(x) = 0$.

4. (a) Suppose that $\sum_{n=0}^{\infty} a_n$ is a convergent series. Define a sequence (b_n) according to $b_n = a_{2n} + a_{2n+1}$ for $n \in \mathbb{N} \cup \{0\}$. Prove that $\sum_{n=0}^{\infty} b_n$ converges.

(b) Construct an example of a series $\sum_{n=0}^{\infty} a_n$ that diverges, but that if (b_n) is defined as above, then $\sum_{n=0}^{\infty} b_n$ converges.

5. Suppose f a is real-valued continuous function on $\mathbb{R}$ and that $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$ where $a < b$. Prove that there exists an $x \in (a, b)$ such that $f(x) = 0$.

6. Let f be a real-valued function defined on an interval $[0, b]$ as

$$f(x) = \begin{cases} x & \text{for } x \in \mathbb{Q}, \\ 0 & \text{for } x \notin \mathbb{Q}. \end{cases}$$

Consider a partition $P = \{0 = t_0 < t_1 < \ldots < t_n = b\}$. What are the upper and lower Darboux sums $U(f, P)$ and $L(f, P)$? Is f integrable on $[0, b]$?

7. Let f be a decreasing function defined on $[1, \infty)$, where $f(x) \geq 0$ for all $x \in [1, \infty)$. Prove that $\int_1^{\infty} f(x)dx$ converges if and only if $\sum_{n=1}^{\infty} f(n)$ converges. *[This is essentially a question asking you to prove a general form of the integral test.]*

8. Consider the function defined for $x, y \in \mathbb{R}$ as

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

(a) Prove that d defines a metric on $\mathbb{R}$.

(b) What is the neighborhood of radius 1/2 centered on 0?

(c) Consider an arbitrary set $S \subseteq \mathbb{R}$. Is S open? Is S compact?

9. Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ be in $\mathbb{R}^2$. Consider the function

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|.$$

(a) Prove that d is a metric on $\mathbb{R}^2$.

(b) Compute and sketch the neighborhood of radius 1 at $(0, 0)$.

10. Consider a function f defined on $\mathbb{R}$ which satisfies

$$|f(x) - f(y)| \leq (x - y)^2$$

for all $x, y \in \mathbb{R}$. Prove that f is a constant function.

11. Suppose that f is differentiable on $\mathbb{R}$, and that $2 \leq f'(x) \leq 3$ for $x \in \mathbb{R}$. If $f(0) = 0$, prove that $2x \leq f(x) \leq 3x$ for all $x \geq 0$.

12. Show that if f is integrable on $[a, b]$, then f is integrable on every interval $[c, d] \subseteq [a, b]$.

13. (a) Suppose r is irrational. Prove that $r^{1/3}$ and $r + 1$ are irrational also.

(b) Prove that $(5+\sqrt{2})^{1/3}+1$ is irrational.

14. By using L'Hôpital's rule, or otherwise, evaluate

$$\lim_{x\to 0}\frac{x}{1-e^{-x^2-3x}}, \qquad \lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right), \qquad \lim_{x\to 0}\frac{x^3}{\sin x - x}.$$

15. Let $a\in\mathbb{R}$. Consider the sequence (s_n) defined as

$$s_n=\begin{cases} a & \text{if } n \text{ is odd,}\\ 2^{-n} & \text{if } n \text{ is even.}\end{cases}$$

Compute $\limsup s_n$ and $\liminf s_n$. For what value of a does (s_n) converge?

16. Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ defined as

$$f(x_1,x_2)=\frac{1}{x_1^2+x_2^2+1}.$$

With respect to the usual Euclidean metrics on $\mathbb{R}$ and $\mathbb{R}^2$, prove that f is continuous at $(0,0)$ and at $(0,1)$.

17. (a) Calculate the improper integral

$$\int_0^1 x^{-p}\,dx$$

for the cases when $0<p<1$ and $p>1$.

(b) Prove that

$$\int_0^\infty x^{-p}dx=\infty$$

for all $p\in(0,\infty)$.

18. Prove that if f is integrable on $[a,b]$, then

$$\lim_{d\to b^-}\int_a^d f(x)\,dx=\int_a^b f(x)\,dx.$$

19. Let $f(x)=x^2$, and define a sequence (s_n) according to $s_1=\lambda$ and $s_{n+1}=f(s_n)$ for $n\in\mathbb{N}$. Prove that (s_n) converges for $\lambda\in[-1,1]$, and diverges for $|\lambda|>1$.

20. Consider the three sets

$$A=[0,\sqrt{2}]\cap\mathbb{Q}, \qquad B=\{x^2+x-1\,:\,x\in\mathbb{R}\}, \qquad C=\{x\in\mathbb{R}\,:\,x^2+x-1<0\}.$$

For each set, determine its maximum and minimum if they exist. For each set, determine its supremum and infimum. Detailed proofs are not required, but you should justify your answers.

21. Let $f_n(x) = x - x^n$ on $[0,1]$ for $n \in \mathbb{N}$.

    (a) Prove that f_n converges pointwise to a limit f, and determine f.

    (b) Prove that f_n does not converge uniformly to f.

    (c) Find an interval I contained in $[0,1]$ on which $f_n \to f$ uniformly.

    (d) Prove that the f_n are integrable, that f is integrable, and that $\int_0^1 f_n \to \int_0^1 f$.

22. Define

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ -2 & \text{if } 1 < |x| \leq 2, \\ 0 & \text{if } |x| > 2 \end{cases}$$

for $x \in \mathbb{R}$.

    (a) Calculate $F(x) = \int_0^x f(t)dt$ for $x \in \mathbb{R}$.

    (b) Sketch f and F.

    (c) Compute F' and state the precise range over which F' exists. You may make use of the second Fundamental Theorem of Calculus.

23. (a) Let f and g be continuous functions on $[a,b]$ such that $\int_a^b f = \int_a^b g$. Prove that there exists an $x \in [a,b]$ such that $f(x) = g(x)$.

    (b) Construct an example of integrable functions f and g on $[a,b]$ where $\int_a^b f = \int_a^b g$ but that $f(x) \neq g(x)$ for all $x \in [a,b]$.

24. Define the sequence of functions h_n on $\mathbb{R}$ according to

$$h_n(x) = \begin{cases} n & \text{if } |x| \leq 1/2n, \\ 0 & \text{if } |x| > 1/2n. \end{cases}$$

    (a) Sketch h_1, h_2, and h_3.

    (b) Prove that h_n converges pointwise to 0 on $\mathbb{R}/\{0\}$. Prove that $\lim_{n\to\infty} h_n(0) = \infty$.

    (c) Let f be a continuous real-valued function on $\mathbb{R}$. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h_n f = f(0).$$

    (d) Construct an example of an integrable function g on $\mathbb{R}$ where

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h_n g$$

    exists and is a real number, but does not equal $g(0)$.

25. Consider the function

$$f(x) = \frac{x}{1+x}.$$

on the interval $[0, \infty)$.

(a) Show that $\lim_{x\to\infty} f(x) = 1$, and that $0 \leq f(x) < 1$ for all $x \in [0, \infty)$.

(b) Sketch f.

(c) Calculate f', f'', and use them to construct the partial Taylor series at $x = 1$ with the form

$$f_T(x) = \sum_{n=0}^{2} \frac{(x-1)^n f^{(n)}(1)}{n!}.$$

(d) Show that f_T can be written as a quadratic equation with the form $ax^2 + bx + c$, and compute a, b, and c.

(e) Add a sketch of f_T to the sketch of f. *[Note: $f_T(1) = f(1)$ so the two curves should intersect at $x = 1$.]*