Math 104: Homework 9 (due April 15)

1. Ross exercise 28.4

2. Suppose that f_n is a sequence of real-valued functions defined on an interval $[a,b]$ that converges uniformly to a function f. Let $x_0 \in [a,b]$, and suppose that

$$\lim_{x \to x_0} f_n(x) = l_n$$

   for all $n \in \mathbb{N}$.

   (a) Prove that l_n is a Cauchy sequence, and hence that it converges to a limit l.

   (b) Prove that $\lim_{x \to x_0} f(x) = l$.

3. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function, where $f(0) = 0$ and $f''(x) \geq 0$ for all $x > 0$. Prove that $f(x)/x$ is increasing for $x > 0$.

   (b) If $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable, $f(0) = 0$ and if $f(x)/x$ is increasing for all $x > 0$, show that $f''(x) \geq 0$ for some $x > 0$, but not necessarily for all $x > 0$. *[Hint: consider $f(x) = x(1 - e^{-x})$.]*

4. Ross exercise 29.17

5. Let $f(x) = |x|^3$. Compute $f'(x)$, $f''(x)$, and show that $f^{(3)}(0)$ does not exist.

6. Ross exercise 30.1

7. Suppose that f is differentiable at a point a. Define

$$\begin{aligned} L_1(a,h) &= \frac{f(a+h) - f(a-h)}{2h}, \\ L_2(a,h) &= \frac{-f(a+2h) + 8f(a+h) - 8f(a-h) + f(a-2h)}{12h}. \end{aligned}$$

   (a) Prove that $\lim_{h \to 0} L_i(a,h) = f'(a)$ for $i = 1, 2$.

   (b) Consider the case when $f(x) = x^5$. How does $|L_i(a,h) - f'(a)|$ behave as $h \to 0$ for $i = 1, 2$? Is there a difference in the rate of convergence?