Math 104: Homework 2 (due February 4)

1. Consider each of the following sets:

$$\begin{aligned} &A = (0,\infty), && B = \{\tfrac{1}{m} + \tfrac{1}{n} \mid m,n \in \mathbb{N}\}, && C = \{x^2 - x - 1 \mid x \in \mathbb{R}\}, \\ &D = [0,1] \cup [2,3], && E = \cup_{n=1}^{\infty} [2n, 2n+1], && F = \cap_{n=1}^{\infty} (1 - \tfrac{1}{n}, 1 + \tfrac{1}{n}). \end{aligned}$$

   For each set, determine its minimum and maximum if they exist. In addition, determine each set's infimum and supremum, writing your answers in terms of infinity for unbounded sets. Detailed proofs are not required.

2. Ross exercise 4.11

3. Ross exercise 4.14

4. Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $M = \{a \cdot b \mid a \in A, b \in B\}$. Is $\sup M = (\sup A) \cdot (\sup B)$? Either prove, or provide a counterexample.

5. Find the limits of each of the following sequences, defined for $n \in \mathbb{N}$:

   (a) $\left(\frac{3n}{n+3}\right)^2$,
   (b) $\frac{1+2+\ldots+n}{n^2}$,
   (c) $\frac{a^n - b^n}{a^n + b^n}$ for $a > b > 0$,
   (d) $n^2/2^n$,
   (e) $\sqrt{n+1} - \sqrt{n}$.

   Detailed proofs are not required, but you should justify your answers.

6. Ross exercise 7.4

7. Let $s_n = n!/n^n$ for all $n \in \mathbb{N}$. Prove that $s_n \to 0$ as $n \to \infty$.

8. Let $(s_n)_{n \in \mathbb{N}}$ be a sequence such that $s_n \to s$ as $n \to \infty$. Prove that if $p : \mathbb{R} \to \mathbb{R}$ is a polynomial function, then $p(s_n) \to p(s)$. *[Hint: Use the limit theorems for addition and multiplication of sequences.]*

9. Let $s_1 = t$ for some $t \in \mathbb{R}$, and define a sequence according to $s_{n+1} = 1 + \frac{s_n}{2}$ for $n \in \mathbb{N}$. Prove that for all t, $s_n \to 2$ as $n \to \infty$.

10. **Optional for the enthusiasts.** Let $s_1 = t$ for $t \in \mathbb{R}$, and define $s_{n+1} = \frac{s_n}{2}(3 - s_n^2)$ for $n \in \mathbb{N}$. For all values of t, determine and prove whether s_n is convergent, divergent but bounded, or divergent and unbounded.