Math 104: Homework 10 (due April 22)

1. Construct a Taylor series expansion for the function $f(x) = \log(1+x)$ at $x = 0$. By considering the remainder $R_n(x)$, prove that the Taylor series agrees with f in the range $-1/2 < x < 1$.

2. Ross exercise 32.7

3. Suppose f is a continuous function on $[a,b]$, and $f(x) \geq 0$ for all $x \in [a,b]$. Prove that if $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a,b]$.

4. Construct an example of a function where $f(x)^2$ is integrable on $[0,1]$ but $f(x)$ is not.

5. Ross exercise 33.7

6. Ross exercise 33.8

7. (a) For any two numbers $u, v \in \mathbb{R}$, prove that $uv \leq (u^2 + v^2)/2$. Let f and g be two integrable functions on $[a,b]$. Prove that if $\int_a^b f^2 = 1$ and $\int_a^b g^2 = 1$ then

$$\int_a^b fg \leq 1.$$

   (b) Prove the Schwarz inequality, that for any two integrable functions f and g on an interval $[a,b]$,

$$\left| \int_a^b fg \right| \leq \left(\int_a^b f^2 \right)^{1/2} \left(\int_a^b g^2 \right)^{1/2}.$$

   (c) Let X be the set of all continuous functions on the interval $[a,b]$. For any $f, g \in X$, define

$$d(f,g) = \left(\int_a^b |f-g|^2 \right)^{1/2}.$$

   Prove that d is a metric.