Mathematics 475 **Fall 1999** **Wilson**

Your Name: ______________________________

Final Exam 12/21/99

- This exam has 8 pages and 12 problems. There are problems on the back of this sheet.
- Where possible please write your answers in the spaces provided right after the problems. If you need more space, either on the exam sheets or on extra paper, please make sure that you have made clear what work goes with what problem!
- There is scratch paper provided on the back of the last sheet.
- You may leave lengthy calculations unevaluated, if you give enough information that any one who can use a calculator with buttons for exponential functions, logarithms, and combinatorial coefficients (as well as ordinary arithmetic) could carry them out.
- How you get an answer is at least as important as a final answer for these problems. Please show all work that could possibly help explain what you are doing, and feel free to write comments on your reasoning. (But don't feel you need to put in irrelevant filler....)

Problem 1 (17 points)

You are going to make a necklace by putting 12 beads on a string which will be tied into a loop: The beads can slide freely around the loop (even over the knot) but one bead cannot pass another. There are 6 red beads and 6 green beads. How many different necklaces can be made? (Two necklaces are counted the same if sliding the beads around the loop can transform one into the other.)

(There is an ambiguity in this question. Answering correctly either version of the question is sufficient. Identifying the ambiguity and answering both versions will get extra credit.)

Problem 2 (17 points)

Show that if $n + 1$ integers are chosen from the set $\{1, 2, 3, \ldots, 3n\}$ then there are always two which differ by at most 2.

Problem 3 (17 points)
Find all graphs of order 5 which have two vertices of degree 1 and no vertices of degree 0. ("Find" means for this purpose draw a representation of each one, up to isomorphism. You do not need to draw multiple copies of a graph with different labelings of the vertices. Also, for this problem, "graph" implies no loops or multiple edges.)

Problem 4 (17 points)
As a collector of classic widgets you have to arrange your collection on two shelves. You will line them up in a row on each shelf. You have four kinds of widgets: There are 10 type A widgets, 8 type B widgets, 6 type C widgets, and 8 type D widgets. Each shelf can hold 16 widgets. Type A widgets all must go on the top shelf. Type D widgets all must go on the bottom shelf. How many different arrangements are possible?
(Widgets line up like books on a shelf, in a row from one end of the shelf to the other. Two ways of putting the widgets on the shelves which differ only by rearranging widgets of the same type, e.g. by shuffling the type A widgets among themselves and similarly for the other types, are counted as being the same. If a particular location is changed to a different kind of widget, however, that does get counted as different.)

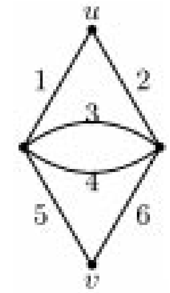
Problem 5 (16 points)
$K_{m,n}$ denotes the complete bipartite graph (no loops or multiple edges) with all possible edges between sets of m and n vertices and no edges within those sets.

(a) What is the degree sequence of $K_{m,n}$ when $m > n$?

(b) What lengths can cycles have in $K_{3,2}$? (Give a list of all the numbers for which there is at least one cycle with that length.)

Problem 6 (17 points)
Find the number of integers between 1 and 10,000 inclusive which are not divisible by 4, 5, or 7.

Problem 7 (16 points)



The Shannon Switching Game is played on the (multi)graph G with selected vertices u and v shown at the right. (The numbers on the edges are just labels.)

(a) Find an induced subgraph with two subtrees T_1 and T_2 having no edges in common.

(b) If player N goes first and marks edge 3 with a $-$ sign, where should player P respond with a $+$ sign? (If there are several appropriate responses show all.)

Problem 8 (16 points)

Suppose you live in a city which is laid out with streets in a rectangular grid, some going north and south and the rest going east and west. The video store is 12 blocks west of your house and 8 blocks north. The pizza place is 5 blocks west and 6 blocks north. When you walk from your house to the video store you always walk as few blocks as possible (20) but you may take different routes.

(a) How many different routes can you take from your house to the video store?

(b) Suppose you want to stop at the pizza place on your way to the video store. Now how many routes are there?

Problem 9 (17 points)
Solve the nonhomogeneous recurrence relation

$$\begin{aligned} h_n &= -2h_{n-1} + 6, \qquad (n \geq 1) \\ h_0 &= 5. \end{aligned}$$

Problem 10 (16 points)
Let $S = \{2,\ 3,\ 4, \ldots,\ 1000\}$. Define two relations on S by (i) $a \diamond b$ if and only if a and b have the same prime factors and (ii) $a \star b$ if and only if each prime factor of a also divides (evenly) into b. Exactly one of these two relations is an equivalence relation.
(A prime factor of an integer is a prime number which divides evenly into the number. E.g., the prime factors of the integer 12 are 2 and 3.)

(a) Which one is an equivalence relation? Show that it is.

(b) Which one is not an equivalence relation? Show that it is not. (I.e., find a specific example of something which must be true for an equivalence relation but which is not true for this relation.)

Problem 11 (17 points)
Prove that for a positive integer n,

$$1+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\cdots+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}.$$

(Hints: use binomial expansion, integrate)

Problem 12 (17 points)
Find the generating function for the sequence

$$0,-1,\ 1,-\frac{1}{2},\ \frac{1}{6},-\frac{1}{24},\ldots,\frac{(-1)^n}{(n-1)!}\ldots.$$

(If you work with this a little you should be able to recognize the Maclaurin series for a function you know.)

SCRATCH PAPER