Mathematics 475 Fall 1999 Wilson

Exam I 10/20/99

Solutions

Problem 1: If we put 5 rooks on a board so that no two attack each other, they must be in 5 different columns and 5 different rows. On a 9×9 board we can pick the 5 columns in $\binom{9}{5}$ ways and the 5 rows also in $\binom{9}{5}$ ways. Making those choices essentially selects a 5×5 "sub-board" on which we put the rooks. Once that sub-board is chosen, there would be 5! ways to put the rooks on it if they were all the same color. (See examples in text, or just think of putting a rook in the top row of the 5, in one of 5 columns, which leaves only 4 columns available for the rook in the second row, etc.) If we think of the rooks in order from the top row to the fifth row, we can assign the colors by picking some 3 places out of 5 to place a blue rook and 2 to place a red one. That can be done in $\binom{5}{3}$ ways. (Once the three places for blue are picked, the red ones are just the ones left over.) Hence the answer is $\binom{9}{5}\binom{9}{5}5!\binom{5}{3} = 19,051,200$ ways.

Problem 2: The total capacity of the shelves is 240 books, and there are 240 books you have to put on them. So arranging them subject to the given constraints amounts to: Pick some 90 of the 120 places on the top shelf for the Math books, and fill the remainder of the places on that shelf with Philosophy books. Pick some 70 of the 120 places on the lower shelf for English books, and fill the remainder of the places on that shelf with Philosophy books. This will complete putting in all of the books and will fill all of the spaces. This can be done in $\binom{120}{90}\binom{120}{70}$ ways. (That is about 3.117×10^{62}.)

Problem 3:

(a) You can use any of several forms of the pigeon hole principle. Thinking through it rather than grabbing for a formula: How many times can we eat there and <u>not</u> have at least one item at least 10 times? We could have the first item 9 times, the second also 9 times, and so on through the fifth item on the menu. That would account for 45 meals. So if we have 46 meals there we must have had at least one item at least ten times.

(b) Since there is no limit on the availability of each item, we could (as an example) have any of the non-pizza items every meal forever. Hence there is no number of meals that guarantees at least ten pizzas.

(c) Let x_i be the number of times you have item i, for $i = 1 \ldots 5$. Then $x_i \geq 0$ for each i, and $x_1 + x_2 + x_3 + x_4 + x_5 = 50$. The number of solutions is then $\binom{5+50-1}{50} = \binom{5+50-1}{4} = \binom{54}{4} = 316,251$.

Problem 4: Since nobody is acquainted with him/herself, the maximum number of acquaintances a person in this group can have is less than 50, and even, so it is 48. Consider two cases: (a) some person has 48 acquaintances, and (b) nobody has 48 acquaintances.

In case (a), suppose a person X has 48 acquaintances. Let S be the set consisting of X and all of his/her acquaintances. Then $|S| = 49$. Each person in S other than X is acquainted with at least one other person because he/she is acquainted with X, and by the even-ness requirement is thus acquainted with at least 2 people. Since X is acquainted with 48 people, any person in S is acquainted with an even number which is at least 2 and at most 48 of people. There are 24 such numbers. Therefore the function that associates with each person in S his/her number of acquaintances takes 49 people to only 24 different numbers, and by the strong form of the pigeonhole principle, some three people must go to the same number.

In case (b) any person has a number of acquaintances which is even and ranges from 0 to 46. There are 24 such numbers. The function which takes a person to his/her number of acquaintances in this case takes 50 people to 24 numbers, so by the same reasoning some three people must go to the same number.

Problem 5: Use the binomial theorem:

$$(2x-4y)^{20} = \sum_{k=0}^{n} \binom{n}{k} (2x)^{n-k}(-4y)^k.$$

(a) To get x^5y^{15} we must have the term with $(2x)^5(-4y)^{15}$, which will include the binomial coefficient $\binom{20}{15}$. Hence the term is $\binom{20}{15}(2x)^5(-4y)^{15} = 2^5(-4)^{15}\binom{20}{15}x^5y^{15}$. Working that out gives $-2^{35}\binom{20}{15} \approx -5.327134 \times 10^{14}$ for the coefficient.

(b) To get x^6y^{14} we must have the term with $(2x)^6(-4y)^{14}$, which will include the binomial coefficient $\binom{20}{14}$. Hence the term is $\binom{20}{14}(2x)^6(-4y)^{14} = 2^6(-4)^{14}\binom{20}{14}x^6y^{14}$. Working that out gives $2^{34}\binom{20}{14} \approx 6.658917 \times 10^{14}$ for the coefficient.

Problem 6: There are many ways to do this. Here is one: Apply the binomial theorem to $(1-x)^n$. (We think of doing this because (i) it will have the alternating signs, (ii) it will have powers of x which when integrated will yield the multipliers $\frac{1}{2}$, $\frac{1}{3}$, etc.) We get

$$(1-x)^n = 1 - \binom{n}{1}x + \binom{n}{2}x^2 - \binom{n}{3}x^3 + \cdots + (-1)^n\binom{n}{n}x^n.$$

Consider both sides as functions of x and integrate: (remember the effect of the $-x$... You may want to make a "u-substitution.")

$$-\frac{1}{n+1}(1-x)^{n+1} = x - \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 - \cdots + (-1)^n\frac{1}{n+1}\binom{n}{n}x^{n+1} + C$$

where C denotes a constant of integration: For any fixed n it is a constant, but it might change with n. To evaluate the constant, let $x = 0$ and get $C = -\frac{1}{n+1}$. Now substitute instead $x = 1$: This gives

$$1 - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \cdots + (-1)^n\frac{1}{n+1}\binom{n}{n} = -C = \frac{1}{n+1}.$$

Problem 7:

(a) There are $\binom{4}{2} = 6$ 2-combinations from a set of 4 elements. They are:

$$\{A,B\}, \{A,C\}, \{A,D\}, \{B,C\}, \{B,D\}, \text{ and } \{C,D\}.$$

(b) The 2-permutations will be just those 2-combinations and their reversals, written so as to indicate that order matters. That will make 12 of them. They are:

$$(A,B), (A,C), (A,D), (B,C), (B,D), \text{ and } (C,D)$$

as given above together with

$$(B,A), (C,A), (D,A), (C,B), (D,B), \text{ and } (D,C).$$