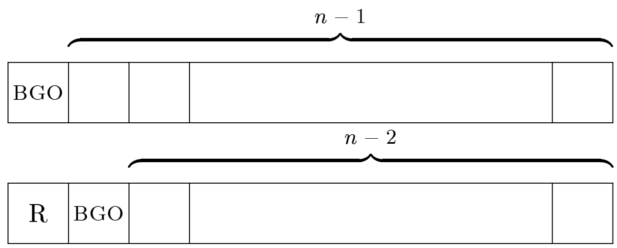
Answers to Math 475 exam II, fall 1999

Problem 1: Let h_n be the number of ways that a $1 \times n$ chessboard can be colored with the four colors Red, Blue, Green, and Orange, in such a way that no two adjacent squares are colored Red. Find a recurrence relation satisfied by h_n.
Consider two possibilities:



The first square of the $1 \times n$ board must either be Red or something else. If it is not Red it must be Blue or Green or Orange. In any of those cases, the remaining $n-1$ squares can be colored in any of the h_{n-1} ways that do not have adjacent Red squares, since the Blue or Green or Orange square cannot combine with the first square of the $n-1$ to produce adjacent Red squares. Since there were three ways to choose the color of the first square other than Red, this yields $3h_{n-1}$ ways to color the board if the first square is not Red.
On the other hand if the first square is Red, the second one must be Blue or Green or Orange to avoid having the first two squares violate the requirement. Then the remaining $n-2$ squares can be colored in any of the h_{n-2} ways that do not have two adjacent Red squares. The three choices for the color of the second square yield $3h_{n-2}$ ways of coloring the board if the first square is Red. Hence the total number is $h_n = 3h_{n-1} + 3h_{n-2}$.

Problem 2: Solve the recurrence relation $h_n = -h_{n-1} + 12h_{n-2}$ (for $n \geq 2$) with the initial values $h_0 = 1$ and $h_1 = -11$.
We assume that h_n can be given as a power of a constant, $h_n = q^n$. The recurrence relation then gives us the equation $q^n + q^{n-1} - 12q^{n-2} = 0$ which we factor as $q^{n-2}(q-3)(q+4) = 0$. The possibility $q = 0$ does not satisfy the initial conditions, so we must have $q = 3$ or $q = -4$. Since this was a linear homogeneous recurrence relation, we can use any linear combination of 3^n and $(-4)^n$ as a solution, i.e. $h_n = c_1 3^n + c_2(-4)^n$ for some constants c_1 and c_2. Using the condition $h_0 = 1$ gives us $1 = c_1 + c_2$ and using $h_1 = -11$ gives $-11 = 3c_1 - 4c_2$. Solving these we get $c_1 = -1$ and $c_2 = 2$. That gives us the solution $h_n = -(3^n) + 2(-4)^n$. You can check this for the first several values of n against the sequence produced by the recurrence relation.

Problem 3: For a holiday gift-giving pool, each of 12 people writes his/her name on a slip of paper and puts it into a hat. The pieces of paper are then mixed up, after which each person draws out one name.

(a) In how many different ways can this happen so that no person draws his/her own name?

This is just the number of derangements D_{12} of 12 items. From the formula we get

$$D_{12} = 12!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{12!}\right).$$

You did not have to evaluate that expression, but it works out to $176,214,841$.

(b) In how many different ways can this happen so that at least one person draws his/her own name?

The total number of ways to distribute the names is 12!. The condition that at least one person gets his/her own name is the same as not having every person get someone else's hat, which part (a) says can be done in D_{12} ways. Hence the total 12! minus the ways we can fail to have someone get his/her own hat is the answer,

$$12! - D_{12} = 479,001,600 - 176,214,841 = 302,786,759.$$

Problem 4: Solve the recurrence relation $h_n = -3h_{n-1} + 4n$ (for $n \geq 1$) with initial condition $h_0 = -\frac{1}{4}$.

This is a linear but not homogeneous recurrence relation. We first solve the homogeneous part, $h_n = -3h_{n-1}$. You can work that out using 'brute force' but it is almost obvious that $h_n = (-3)^n$ is a solution and hence the general solution is $c(-3)^n$. Now we need to find one solution to the real, nonhomogeneous, relation. Since the nonhomogeneous part $4n$ is of the form $an+b$ we try to find a solution of the form $h_n = rn + s$ for some constants r and s. Putting that form into the recurrence relation we have $rn+s = -3[r(n-1)+s]+4n = (-3r+4)n+3(r-s)$. Equating the coeficients on n we get $r = -3r+4$ which we solve to get $r = 1$. Equating the constant terms we have $s = 3r - 3s$ which we solve (using $r = 1$) to get $s = \frac{3}{4}$. Hence the general solution to the recurrence relation is $h_n = c(-3)^n + n + \frac{3}{4}$. Now we use the condition that $h_0 = -\frac{1}{4}$ to get $-\frac{1}{4} = c + \frac{3}{4}$ and hence $c = -1$. That gives the solution to the recurrence relation which has the correct value of h_0 as

$$h_n = -(-3)^n + n + \frac{3}{4}.$$

Problem 5: Count the permutations $i_1 i_2 i_3 i_4 i_5$ of $\{1,\ 2,\ 3,\ 4,\ 5\}$ which satisfy $i_1 \notin \{2,\ 3\}$, $i_2 \notin \{2,\ 3\}$, $i_3 \notin \{4,\ 5\}$, and $i_4 \notin \{4,\ 5\}$. (There are no restrictions on i_5.)

I find it convenient to think of this as a rook-arrangement problem on the 5×5 board with forbidden squares: See Figure 1. Using the notation from the text, let r_k be the number of ways to put k non-attacking rooks on that 5×5 board where each of the k rooks is in a forbidden position. Then we know the answer will be $5! - r_1 \cdot 4! + r_2 \cdot 3! - r_3 \cdot 2! + r_4 \cdot 1! - r_5 \cdot 0!$. Since we cannot put more than 4 of the rooks in forbidden positions (corresponding to the fact that there is no restriction on i_5), $r_5 = 0$ and we can ignore the last term in that sum.

Now we need to work out the values for r_k. We can put one rook on the board (of course just one will be "non-attacking") in forbidden squares in 8 ways since there are 8 forbidden squares. Thus $r_1 = 8$.

To put two non-attacking rooks on the board, sticking to the forbidden squares, we could (i) have them both in the upper 2×2 cluster of X's, (ii) have them both in the lower cluster, or (iii) have

	X	X		
	X	X		
			X	X
			X	X

Figure 1: Chess Board for Problem 5

one in each. To put them both in the upper (or lower) cluster without attacking each other they must be on the diagonals of that cluster, so there are two ways to do (i) and two ways to do do (ii). For one in each we can have the upper one in any of four places and the lower one in any of four places, for a total of 16 ways to do (iii). Thus $r_2 = 2 + 2 + 16 = 20$.
For three rooks they can be (i) two in the upper and one in the lower or (ii) vice versa. We can do (i) in $2 \cdot 4$ ways, two ways of putting two rooks in the upper cluster without attacking each other and four ways to choose the square for the third rook in the lower cluster. The number of ways to do (ii) is the same, we just interchange the roles of the two clusters. Hence $r_3 = 8 + 8 = 16$.
For four rooks we must put two in each cluster: We can arrange the two in the upper cluster in either of two ways, and likewise for the other two in the lower cluster, so $r_4 = 2 \cdot 2 = 4$.
Putting these into the formula above, the answer is

$$5! - 8 \cdot 4! + 20 \cdot 3! - 16 \cdot 2! + 4 \cdot 1! = 20.$$

Problem 6: Find a generating function $g(x)$ for the recurrence relation $h_n = -4h_{n-1} + 3h_{n-2}$ (for $n \geq 2$) with initial conditions $h_0 = 1$ and $h_1 = -1$. (You do not need to find a series representation for $g(x)$, you do not need to decompose $g(x)$ using partial fractions, and you do not need to solve for h_n as an explicit function of n.)
Assume that $g(x)$ is a generating function for h_n. Then

$$\begin{array}{rclllllll} g(x) & = & h_0 & +h_1 x & +h_2 x^2 & +\ldots & +h_n x^n & +\ldots \\ 4xg(x) & = & & 4h_0 x & +4h_1 x^2 & +\ldots & +4h_{n-1}x^n & +\ldots \\ -3x^2 g(x) & = & & & -3h_0 x^2 & -\ldots & -3h_{n-2}x^n & -\ldots \end{array}$$

and if we add these lines we get $g(x)(1 + 4x - 3x^2) = h_0 + h_1 x + 4h_0 x$. (All other terms cancel out, e.g. $h_2 x^2 + 4h_1 x^2 - 3h_0 x^2 = x^2(h_2 + 4h_1 - 3h_0) = 0$ by the recurrence relation.) Using the values of h_0 and h_1 we get $g(x)(1 + 4x - 3x^2) = 1 - x + 4x = 1 + 3x$. Dividing,

$$g(x) = \frac{1 + 3x}{1 + 4x - 3x^2}.$$

Problem 7: Find the number of 12-combinations of the multiset

$$\mathcal{T} = \{3 \cdot a, 4 \cdot b, 5 \cdot c, 4 \cdot d\}.$$

Let $\mathcal{T}^*$ be the multiset with infinitely many copies of a, b, c, and d, $\mathcal{T}^* = \{\infty \cdot a, \infty \cdot b, \infty \cdot c, \infty \cdot d\}$. Using the notation in the text, let S be the set of all 12-combinations of $\mathcal{T}^*$. Let A_1 be the set of all 12-combinations of $\mathcal{T}^*$ with more than 3 a's, A_2 the ones with more than 4 b's, A_3 the ones with more than 5 c's, and A_4 the ones with more than 4 d's. Then by inclusion/exclusion the number we want can be found as

$$|S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + \quad etc.$$

Using a formula from earlier in the course, $|S| = \binom{12+4-1}{12} = \binom{15}{12}$. Any combination in A_1 contains at least 4 a's so by removing 4 a's from the combination we get an 8-combination of $\mathcal{T}^*$. Thus $|A_1| = \binom{8+4-1}{8} = \binom{11}{8}$. In the same way we get $|A_2| = \binom{7+4-1}{7} = \binom{10}{7}$, $|A_3| = \binom{8+4-1}{6} = \binom{9}{6}$, and $|A_4| = \binom{7+4-1}{7} = \binom{10}{7}$.

Any 12-combination in $A_1 \cap A_2$ has at least 4 a's and at least 5 b's. Removing 4 a's and 5 b's gives a 3 combination of $\mathcal{T}^*$. Hence $|A_1 \cap A_2| = \binom{3+4-1}{3} = \binom{6}{3}$. Similarly we get for the other pairs $|A_1 \cap A_3| = \binom{2+4-1}{2} = \binom{5}{2}$, $|A_1 \cap A_4| = \binom{3+4-1}{3} = \binom{6}{3}$, $|A_2 \cap A_3| = \binom{1+4-1}{1} = \binom{4}{1}$, $|A_2 \cap A_4| = \binom{2+4-1}{2} = \binom{5}{2}$, and $|A_3 \cap A_4| = \binom{1+4-1}{1} = \binom{4}{1}$.

Any intersection of three or more of the A_i's is empty: Consider the case of $A_1 \cap A_2 \cap A_4$. A 12 combination in that set would have to have at least 4 a's, at least 5 b's, and at least 5 d's, requiring it to contain at least $4 + 5 + 4 = 13$ elements, which can't happen in a 12-combination. The other cases have even larger numbers involved. So now we have all the numbers needed to work out the expression above, and the answer is

$$\binom{15}{12} - \left[\binom{11}{8} + \binom{10}{7} + \binom{9}{6} + \binom{10}{7}\right] + \left[\binom{6}{3} + \binom{5}{2} + \binom{6}{3} + \binom{4}{1} + \binom{5}{2} + \binom{4}{1}\right].$$

Again you did not have to evaluate that expression but it can be done,

$$= 455 - (165 + 120 + 84 + 120) + (20 + 10 + 20 + 4 + 10 + 4) = 34.$$