Mathematics 443 (Wilson) **Final Exam** **May, 1997**

Your answers for these nine problems must be turned in by 10 AM on Tuesday, May 13. Be sure to put your name on your work and to observe the following conditions:

- You may use your textbook, class notes, computer software such as MATLAB, and/or a powerful calculator.
- Please do not discuss your work with others: If you have questions concerning the exam call me (262-5446 or 263-5944 on campus, 274-3746 at home) or come to see me or contact me by email (wilson@math.wisc.edu).
- While you are welcome to use a computer or calculator, you should write out in your solutions how to do the problems with sufficient detail that someone following your instructions could get the answers without a calculator or computer.

Problem 1:

Consider the augmented matrix for a system of equations given by $\begin{bmatrix} k-3 & 3 & -1 & 0 \\ 6 & k & 2 & 6 \\ 6 & -4 & 2 & 2 \end{bmatrix}$.

For some value(s) of k the system has no solutions, for some value(s) it has exactly one solution, and for some value(s) it has infinitely many solutions. Tell what numbers k produce each situation. For the case where there is one solution, also give that solution. For the case where there are infinitely many solutions, find a form for those solutions in terms of arbitrary parameters.

Problem 2:

Find the LU decomposition of the matrix $\begin{bmatrix} -3 & 7 & 3 & 2 \\ -8 & 5 & 5 & 1 \\ -11 & 0 & 7 & -1 \\ -2 & -4 & 2 & -2 \end{bmatrix}$.

You should show the matrices L, U, and P referred to in Theorem 3.53(a) on page 121 in the text.

Problem 3:

Apply the traditional Gram-Schmidt process (page 231 in text) to the ordered set of vectors $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

Problem 4:

Let B be the ordered basis $\{x^2+1, x-3, x^2+x+1\}$ for $\mathcal{P}^3$. Find a formula for the coordinate isomorphism $c_B(p)$ from $\mathcal{P}^3$ to $\mathcal{R}^3$ as it is applied to a typical member $p = ax^2+bx+c$ of $\mathcal{P}^3$.

Problem 5:

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$. Compute the norms $\|A\|_1$ and $\|A\|_\infty$ using Theorem 6.23 from page 266 in the text. Compute the norm $\|A\|_2$ directly from the definition (page 265). (Hint: Any vector in $\mathcal{R}^2$ can be written as $x = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$ for some r and θ. Calculate $\frac{\|Ax\|_2}{\|x\|_2}$ and use calculus to find what direction θ maximizes this.) Show that your answers satisfy all three inequalities in part (b) of Theorem 6.25.

Problem 6:

Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 5 & 4 & 6 \\ -5 & -1 & -3 \end{bmatrix}$. Find the eigenvalues of A. For each eigenvalue find its algebraic multiplicity, its geometric multiplicity, and a maximal set of linearly independent eigenvectors.

Problem 7:

Let $A = \begin{bmatrix} 9 & -5 & -6 \\ 5 & -2 & -5 \\ 6 & -5 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix}$. For each of A and B tell whether it is possible to find a similarity taking the matrix to a diagonal matrix. If it is, show the similarity by finding matrices P and Λ such that Λ is diagonal and $P^{-1}AP = \Lambda$ (or $P^{-1}BP = \Lambda$). If it is not possible tell how you know that.

Problem 8:

Find a Householder matrix H_w such that $H_w \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ is a multiple of $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$.

Problem 9:

Find both the unnormalized and normalized QR decompositions of $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 1 & 4 & 6 \end{bmatrix}$.