Mathematics 431 (Wilson)

Problem 1: If I describe the outcomes in the form (r, b) to mean a red ball is drawn from urn A and a black one from urn B, the event $E = \{(r, b), (b, r)\}$ is the event that the two balls are different in color. The outcome (r, b) has probability $(5/11) \times (6/13) = 30/143$ since the draws from the two urns do not interact and the probability of drawing red from 5 red and 6 black is 5/11. etc. Similarly the outcome (b, r) has probability $(6/11) \times (7/13) = 42/143$. The two outcomes (considered as events) are mutually exclusive events and so we can add their probabilities to get $P(E) = 72/143 \approx .5035.$

Problem 2: This is a partitioning problem. You could phrase it in terms of numbers, although you don't need to, as "how many ways can we pick 6 numbers (the number of marbles in each box) which add up to 30?" In (a) the numbers can be zero, so the number of ways is $\binom{n+r-1}{r-1} = \binom{35}{5} = 324,632$. In (b) the numbers cannot be zero, so the number of ways is $\binom{n-1}{r-1} = \binom{29}{5} = 118,755$.

Problem 3: The two outcomes b-w-b and w-b-w can be worked out separately and the results combined:

To get b-w-b we must pick black the first time, which has probability 9/16 and changes the urn to 8 white, 9 black. Then we must pick white, which now has probability 8/17. That will change the urn to 8 white and 10 black. Lastly we must pick black again, which now has probability 10/18. Hence the probability of b-w-b is $(9/16)(8/17)(10/18) = \frac{9.8 \cdot 10}{16 \cdot 17 \cdot 18}$. A similar analysis gets $\frac{7 \cdot 10.8}{16 \cdot 17 \cdot 18}$ as the probability of w-b-w.

We can add the probabilities of these mutually exclusive events to get $40/153 \approx .2614$ as the answer.

Problem 4: The problem just asks how may ways you can choose the questions to answer, not how many different orders you could write the answers in. Hence (a) just needs the number of ways of choosing 5 (or 15) items out of 20, $\binom{20}{5} = 15,504$. For (b) we can find the number of ways of choosing 7 out of the first 10 questions and separately the number of ways of choosing 8 out of the last 10, and multiply the results. This gives $\binom{10}{7} \times \binom{10}{8} = 5400$. (As a partial check, note that the restrictions in (b) ought to reduce the number of possibilities, and our answers are consistent with that.)

Problem 5: The experiment essentially amounts to selecting N two-digit integers randomly. There are 100 such numbers, from 0 (or 00) to 99. Rather than directly computing P(some two answers are the same) we compute P(no two answers are the same) and subtract from 1. If N = 1 the probability that no two are the same is clearly 1.

If N = 2: There are 100 choices for the second number, 99 of which are different from whatever the first number turned out to be, so the probability they are not the same is 99/100.

If N = 3: There is still a probability of 99/100 that the second number misses the first. If it doesn't then the we don't have "no two answers the same" no matter what the third number is. If the first two are different, there are 98 numbers left that the third number could be without equalling either of the first two. Hence the probability that all three are different is (99/100)(98/100).

Continuing in this way, for N numbers in general, we will have N-1 such factors and the probability that no two match is $(99/100)(98/100)(\dots)((100 - (N - 1))/100) = \frac{99\cdot98\cdot97\cdots(100 - (N - 1))}{100\cdot100\cdots100}$ $= \frac{99.98.97\cdots(100-(N-1))}{100^{N-1}}.$

(There are several other forms in which this part of the answer can be written: Multiplying by 100/100 one can write it as $\frac{100\cdot99\cdot98\cdots(100-(N-1))}{100^N}$. That in turn can be rewritten as $100\cdot99\cdot98\cdots(100-(N-1))\frac{1}{100^N} = \frac{100!}{(100-N)!}\frac{1}{100^N} = {\binom{100}{N}}\frac{1}{100^N}$, with the caution that this version only applies if $N \leq 100$. (If N > 100 this version will try to calculate the factorial of a negative number...) The previous versions will correctly compute that the probability that all N are distinct is zero if N > 100.) Putting it all together, the answer to the problem is obtained by subtracting this from 1 to get $1 - \frac{99\cdot98\cdot97\cdots(100-(N-1))}{100^{N-1}}$. (For whatever interest it may present, this probability is just barely less than 0.5 for N = 12 and significantly greater than 0.5 for N = 13, and if there are 28 people in our class the probability is about 0.985 that some two have the same last two digits in their Social Security numbers.)

Problem 6: The probability that the ball from urn B is black given that two of the balls are red, $P(\text{ball from urn B is black} \mid \text{two balls are red})$, is P(urn B black AND two red)/P(two red). We calculate first the denominator of this fraction: The three ways to get two reds can be symbolized as BRR, RBR, and RRB. $P(\text{BRR}) = \frac{6\cdot4\cdot7}{9\cdot11\cdot14}$. $P(\text{RBR}) = \frac{3\cdot7\cdot7}{9\cdot11\cdot14}$. $P(\text{RRB}) = \frac{3\cdot4\cdot7}{9\cdot11\cdot14}$. Hence the denominator of the fraction is the sum of those numbers, 399/1386. To calculate the numerator: If the ball from urn B is black and two are red, those two must be the balls from A and C. Hence the event "urn B black AND two red" is the same as RBR, whose probability we set up above. Working that out gives 147/1386. Hence the fraction above works out to $147/399 \approx .368$.

(If you interpreted "two of the chosen balls were red" as meaning at least two, the denominator above has to have added the probability of RRR. The result is $147/1483 \approx .104$.)

Problem 7: Let N be the event that a chosen person reads the newspaper, and T the event that a chosen person watches TV.

(a) The event that a person neither watches TV or reads the newspaper is $(N \cup T)^c$. $P(N \cup T) = P(N) + P(T) - P(N \cap T) = .4 + .7 - .2 = .9$, so $P((N \cup T)^c) = 1 - .9 = .1$. Thus 10% of the population neither watch TV nor read the newspaper. (b) $P(T|N) = P(T \cap N)/P(N) = .20/.40 = .50$.

Problem 8:

(a) One way to describe the sample space is to list the outcomes as $\{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, T)\}$ the outcomes listed are not just the final step in the experiment but the whole observable result from the experiment. One could also list the results as $\{1, 2, 3, 4, 5, 6, H, T\}$, which is more compact but doesn't make it as easy to see what probability to assign to each outcome.

(b) There should be eight outcomes no matter how you choose to denote them.

(c) No: For example the outcome (H, 1) has probability $(1/2) \times (1/6) = 1/12$ while the outcome (T, H) has probability $(1/2) \times (1/2) = 1/4$.

Problem 9: While this problem can be done using only algebraic symbolism it is easier to see what is happening if you draw a picture something like this:



The easiest way to fill in those numbers is to start with the 4% common to all three regions. Then each of the regions common to just two can be filled in, for example 19% in the intersection of vanilla and cake must include 4% common to all three, so we can fill in 15% for the region representing "vanilla and cake but not strawberry". Then continue outward to the regions representing just one purchase. For example, 31% buy vanilla but that includes the ones who buy something else so we subtract 2% (strawberry and vanilla but not cake) and 4% (all three) and 15% (vanilla and cake but not strawberry) to get 10% for the people buying only vanilla.

Once those preliminary results have been worked out the problem is easy to finish: For (a) add the "exactly two" regions to get 2%+7%+15%=24%, or a probability of 0.24. For (b) add all the numbers which are inside any circle to get 0.70.