Problem 1:
Let V be the set of three-element column vectors of real numbers, with the usual addition but a different scalar multiplication. Specifically,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} \quad \text{and} \quad (\text{for any number } c) \quad c \odot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Is V a real vector space? If not, show how it fails to satisfy at least one of the properties required in the definition of a vector space. If it is, show how you know those properties are all satisfied.

Solution:

We can go through the definition's requirements one after another to see if they are all met. Since these definitions did not change from the usual additoin on $\mathbb{R}^3$ which is a vector space, we will not find any problems with the requirements that do not mention the scalar multiplication operation $\odot$. Hence we can start with the requirement that V with $\odot$ be closed: Since any number times a vector gives back the original vector this is surely closed. In fact this set and operations satisfies all of the defining requirements except number (6). To check (6) we have to see whether $(c+d)\odot\vec{u} = c\odot\vec{u}\oplus d\odot\vec{u}$ for every choice of the numbers c and d and every vector $\vec{u}$ in V. Writing out what the two sides of that equation amount to with the given definitions of $\oplus$ and $\odot$ we have $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \oplus \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 2u_1 \\ 2u_2 \\ 2u_3 \end{bmatrix}$. Are these equal? (**) They can be, for example when $u_1 = u_2 = u_3 = 0$, but they are not equal for all possible vectors $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in V$ and hence we do not have a vector space.

(It would suffice to give one example, e.g. $(2+3) \odot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, that does not work...)

(Many people got to the point I marked above as (**) and said the two quantities are not equal. You really should say that they may not be equal, that there are choices for $\vec{u}$, c, and d that make them unequal, since there are some values for which they really are equal, but I let that go....)

Problem 2:
The vectors in P_3

$$\begin{aligned} v_1 &= t^3 + 2t + 3, \\ v_2 &= t^2 + 2, \\ v_3 &= t^3 + 2t^2 + 2t + 7, \text{ and} \\ v_4 &= 2t + 1 \end{aligned}$$

are not linearly independent.

Let W be the subspace of P_3 spanned by $\{v_1,\ v_2,\ v_3,\ v_4\}$.

(a) Find a basis for W.

(b) What is the dimension of W?

Solution:

The short way: Note that $v_3 = v_1 + 2v_2$, so we don't need v_3 to get the same span as with all four. But the set v_1, v_2, v_4 is linearly independent, so we can use that as our basis.

Once you have a basis, the dimension of W is the number of things in that basis and so dim(W)= 3. Many people elected to use matrices to solve this provblem, and maybe 50% of those who so did got it right. If you consider W as the set of all linear combinations of the vectors v_1, v_2, v_3, v_4, we can exprfess that as all possible combinations $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$ and that amounts to $(a_1 + a_3)t^3 + (a_2 + 2a_3)t^2 + (2a_1 + 2a_3 + 2a_4)t + (3a_1 + 2a + 2 + 7a_3 + a_4)$. We need to reduce the set of vectors until this kind of expression has only one solution (the a_i's) when it has any solution at all. If we write the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 2 & 2 \\ 3 & 2 & 7 & 1 \end{bmatrix},$$

then row reduce it to get

$$A_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that the third column of A_R has no leading entry and can be given arbitrary values in the solutions, but all other columns do have leading entries. Hence we can find a basis for W by using the first, second, and fourth of the original vectors.
Note that W is a set of polynomials: A particularly egregious error was to give as a basis for W a column vector or a set of column vectors.

Problem 3:

Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}$.

(a) Find A^{-1}.

(b) Write A as a product of elementary matrices. (Be careful not to confuse A with A^{-1}.)

<u>Solution:</u>
The straight forward way to find A^{-1} is to write the matrix $[A|I_2]$ and apply row operations to get the first two columns to be I_2, at which point the last two will be A^{-1}. But we will also keep track of the elementary row operations applied, for use in (b).
Start with $\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right]$. If we swap the two rows (operation 1) we get $\left[\begin{array}{cc|cc} 2 & 4 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$. Now we can multiply the first row by $\frac{1}{2}$ (operation 2) to get $\left[\begin{array}{cc|cc} 1 & 2 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & 0 \end{array}\right]$. Now if we add (-2) times the second row to the first we get $\left[\begin{array}{cc|cc} 1 & 0 & -2 & \frac{1}{2} \\ 0 & 1 & 1 & 0 \end{array}\right]$. So we have $A^{-1} = \begin{bmatrix} -2 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$.
Now we know that if E_i is the elementary matrix corresponding to operation i above, for $i = 1$ and 2 and 3, $E_3E_2E_1A = I_2$. If we multiply on the left first by E_3^{-1}, then by E_2^{-1}, and then by E_1^{-1}, we get $A = E_1^{-1}E_2^{-1}E_3^{-1}I_2$. Since the inverse of an elementary matrix is just the elementary matrix corresponding to the "undoing" operation, we have $E_3^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, and $E_2^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, and $E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hence $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The answer for A^{-1} is the only correct answer, but you could have different expressions for A as a product of elementary matrices depending on what sequence of operations you applied. But (i) the matrices must be elementary, i.e. correspond to a single row operation, and (ii) their product must give A. The order matters, as usual with matrix products. Quite a few people also included a copy of I_2 either at the beginning or end of the product: That is not necessary but since I_2 is an elementary matrix it is OK.

Problem 4

Let V be the vector space consisting of all continuous functions defined on the real numbers and taking real values, with the usual way of adding functions or multiplying a function by a number.

Let W be the set of those functions in V which have an integral on the interval $[0, 1]$ and that integral produces the number 0, i.e.

$$W = \left\{ f \mid f(x) \text{ is continuous at every } x \text{ and } \int_0^1 f(x)\, dx = 0 \right\}.$$

Show that W is a subspace of V.

Solution:

We need to show three things: (a) $W \neq \emptyset$; (b) If f and g are any two functions satisfying the requirements to be in W then so is $f + g$; (c) If f is any function in W and c is any real number, then cf meets the requirements to be in W.

(a) You can exhibit any function whose integral from 0 to 1 gives zero to show $W \neq \emptyset$. In class I used the functions $\sin(2\pi x)$ and $\cos(2\pi x)$ as members of this set. An even easier example is the function $f(x) = 0$, the function that gives the number 0 for every input x, whose integral is certainly 0.

(b) Suppose f and g are any two functions in W. Then will be continuous on $\mathbb{R}$ and take values in $\mathbb{R}$, because they are also in V, but that is not the important part. We have to show that the function $f + g$ which is their sum again meets the condition to be in W which is essentially that

$\int_0^1 (f + g)(x) dx = 0$. But $(f + g)(x) = f(x) + g(x)$ for any x, so

$$\int_0^1 (f + g)(x) dx = \int_0^1 (f(x) + g(x)) dx = \int_0^1 f(x)\, dx + \int_0^1 g(x)\, dx = 0 + 0 = 0,$$

so $f + g$ does meet the condition and is in W.

(c) For any function f in W and any real number c,

$$\int_0^1 (cf)(x) dx = \int_0^1 c(f(x)) dx = c \int_0^1 f(x)\, dx = c \cdot 0 = 0,$$

so cf meets the condition also and hence is in W.

Comments: Many people seemed to think all the vectors (functions) in W are themselves 0. Since the first two functions suggested as examples in part (a) of the answer were not the zero function, that is clearly wrong. Rather, for a function (zero or not) to be in W its integral from 0 to 1 must give zero. And that zero is the number zero, not a function and hence not to be denoted as a vector in this space. Also, many people overlooked the need to show that W is not empty.

Problem 5:

Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Find a basis for $\mathbb{R}^3$ which includes v_1 and v_2.

Be sure to explain your reasoning. You should indicate how you know the vectors you come up with are a basis for $\mathbb{R}^3$.

Solution:

Since the dimension of $\mathbb{R}^3$ is three, we know we need to have three vectors in any basis. Since we are given two vectors (which are linearly independent, I did not trick you) to be in the basis, we need to find a third vector which I will call $v_3 \in \mathbb{R}^3$ such that $\{v_1,\ v_2,\ v_3\}$ is a basis for $\mathbb{R}^3$. By theorems in the book, given that the dimension of the space is three, if we have three vectors that are linearly independent they must space and vice versa. So all we need to do is find a vector for v_3 such that the three vectors are linearly independent, or that they span.

The quickest way to do this is just to try some vectors until you find one that works: The odds are extremely good that any vector you might pick will work! E.g. if we let $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and test them, the three are linearly independent: One way to do that is to write the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ whose columns are the three vectors, row reduce it, and note that we get I_3. So $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is one correct answer.

For a more systematic approach: Remember how the book created a basis from a spanning set by eliminating redundant vectors from the right hand end. Create the (ordered) set

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$ by adding on the three standard basis vectors.

(You could use any other spanning set instead of those last three columns.) Since the last three by themselves span $\mathbb{R}^3$, certainly the set of all five does. Now make up the 3×5 matrix with those columns, and row reduce. The first, second, and fourth columns will contain leading entries, while the third and fifth do not. Hence we leave out the third and fifth, and use the first, second, and fourth of those vectors as our basis. Note that this is a different correct answer from the one found above.

Problem 6:

Let $A = \begin{bmatrix} 2 & 1 & 3 & 0 & -6 \\ 1 & 0 & 2 & 1 & -1 \\ 1 & 1 & 1 & 0 & -3 \\ 0 & -2 & 2 & 0 & 0 \end{bmatrix}$.

Find the Reduced Row Echelon Form matrix A_R which is row equivalent to A.

Solution:

There is not really a lot to say about this one. You could use different sequences of row operations. I did the following: First I interchanged the first and second rows, getting

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 2 & 1 & 3 & 0 & -6 \\ 1 & 1 & 1 & 0 & -3 \\ 0 & -2 & 2 & 0 & 0 \end{bmatrix}.$$

Then I subtracted twice the first row from the second, and also subtracted the first row from the third, to get

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & -2 & -4 \\ 0 & 1 & -1 & -1 & -2 \\ 0 & -2 & 2 & 0 & 0 \end{bmatrix}.$$

Then I subtracted the second row from the third, and added twice the second row to the last row, giving

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & -2 & -4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -4 & -8 \end{bmatrix}.$$

Next add four times the third row to the last row to get

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & -2 & -4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Lastly, Subtract the third row from the first and add twice the third row to the second to get

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 2 & 0 & -3 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ which is in RREF.}$$

Problem 7:
Consider the system of equations

$$\begin{aligned} x_1 - 3x_3 + 4x_4 &= -3 \\ 2x_2 + 4x_3 - 2x_4 &= 4 \\ 2x_1 + x_2 - 4x_3 + 7x_4 &= -4 \end{aligned}$$

Find all solutions of this system. Express your answer as a combination of certain vectors, some multiplied by arbitrary constants.

Solution:

Start by writing the augmented matrix $[A|B] = \left[\begin{array}{cccc|c} 1 & 0 & -3 & 4 & -3 \\ 0 & 2 & 4 & -2 & 4 \\ 2 & 1 & -4 & 7 & -4 \end{array}\right]$.

Now row reduce that to get $\left[\begin{array}{cccc|c} 1 & 0 & -3 & 4 & -3 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

Notice that the first two columns contain leading entries while the other columns from A_R, the third and fourth columns, do not. We make up three 4-element vectors: One will contain the constants from the last column, what was left of B after row reduction. The other two will be constructed from the two columns that do not contain leading entries. Those two columns are 3 and 4: Fill in 0's in those places in the column coming from B, and fill in 0,1 in those places in one of the other columns and 1,0 in the same places in the remaining column to get $\begin{bmatrix} \\ \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} \\ \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} \\ \\ 0 \\ 1 \end{bmatrix}$ with the top two spaces in each yet to be filled in. Fill in the first from the B column, the second from column 3 with the signs changed, and the last from column 4 with the signs changed, and put multipliers in front of the last two, to get the representation of all solutions as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}. \text{ where } \alpha_1 \text{ and } \alpha_2 \text{ are arbitrary constants.}$$

There were a couple of mistakes made often enough to need comment: (i) Some people left the B part out entirely, or went to the other extreme and used the last column in A_R as if it related to coefficients on a fifth unknown. (ii) Quite a few people wrote the answer as a combination of some vectors of one size and some of another, which of course could not be added!

Problem 8:

Let $A = \begin{bmatrix} -3 & -5 \\ 2 & 4 \end{bmatrix}$.

For what numbers λ will the null space of $\lambda I_2 - A$ contain non-zero vectors?

Solution:

Computing $\lambda I_2 - A$ we get $\begin{bmatrix} \lambda+3 & 5 \\ -2 & \lambda-4 \end{bmatrix}$ (**). If $\lambda + 3 \neq 0$ we can use row operations to convert that to

$$\begin{bmatrix} 1 & \frac{5}{\lambda+3} \\ -2 & \lambda-4 \end{bmatrix} \text{ and then to } \begin{bmatrix} 1 & \frac{5}{\lambda+3} \\ 0 & (\lambda-4)+\frac{10}{\lambda+3} \end{bmatrix}.$$

Now we see there will be non-zero solutions if and only if $(\lambda - 4) + \frac{10}{\lambda+3} = 0$.

(If $\lambda = 3$ the matrix (**) above becomes $\begin{bmatrix} 0 & 5 \\ -2 & -7 \end{bmatrix}$ which produces no non-zero solutions.)

Rewriting that as $(\lambda-4)(\lambda+3) = -10$ or $\lambda^2 - \lambda - 2 = 0$, we find the solutions are $\lambda = -1$ and $\lambda = 2$. We can check those: For $\lambda = -1$, (**) becomes $\begin{bmatrix} 2 & 5 \\ -2 & -5 \end{bmatrix}$, which clearly row reduces to $\begin{bmatrix} 1 & \frac{5}{2} \\ 0 & 0 \end{bmatrix}$ and so has non-zero solutions. For $\lambda = 2$, (**) becomes $\begin{bmatrix} 5 & 5 \\ -2 & -2 \end{bmatrix}$, which similarly has non-zero solutions.