Your Name: ____

Mathematics 340, Spring 2007 **Lecture 1 (Wilson)**

Final Exam May 14, 2007

Write your answers to the twelve problems in the spaces provided. If you must continue an answer somewhere other than immediately after the problem statement, be sure (a) to tell where to look for the answer, and (b) to label the answer wherever it winds up. In any case, be sure to make clear what is your final answer to each problem.

There is a problem on the back of this sheet: Be sure not to skip over it by accident!

There is scratch paper at the end of this exam. If you need more scratch paper, please ask for it.

You may refer to notes you have brought with you, as described in email to the class.

BE SURE TO SHOW YOUR WORK, AND EXPLAIN WHAT YOU DID. YOU MAY RECEIVE REDUCED OR ZERO CREDIT FOR UNSUBSTANTIATED ANSWERS. If you are citing a theorem or other result from the book, you do not need to give theorem number or page, but you should make clear exactly what the cited result says. Don't invent new things in the book that make a problem trivial.

If a problem refers to a vector, like $\vec{v}$ or $\vec{v}_1$, it can be in any vector space unless there is a particular space specified. In any one problem you may assume all vectors come from the same space unless specified otherwise.

The symbol $\mathbb{R}$ is used to denote the set of all real numbers. $\mathbb{R}^n$ and $\mathbb{R}_n$ denote, respectively, the sets (real vector spaces) of all n-element column vectors and of row vectors.

Problem	Points	Score
1	15	
2	16	
3	18	
4	16	
5	18	
6	15	
7	16	
8	18	
9	17	
10	17	
11	16	
12	18	
TOTAL	200	

Problem 1 (15 points)

(a) There are three kinds of elementary row operations and associated elementary matrices. Describe what each kind of operation does when applied to a matrix.

(b) Why is every elementary matrix non-singular? Give a proof. Your proof should make use of part (a) and should not make use of the determinant.

(c) Prove that every non-singular matrix can be written as a product of elementary matrices. You may assume that the inverse of an $n \times n$ matrix A can be found, if it exists, by row-reducing an $n \times 2n$ matrix composed of A and I_n side-by-side.

Problem 2 (16 points)

For each of the following sets and operations, tell whether it is or is not a real vector space. If it is not, show a property a vector space must have that this one fails. (Be specific and make clear why this example fails to have that property.) f it is a vector space, show how you know it has the closure properties for addition and scalar multiplication (i.e., that the sum and scalar product defined for it always produce results within the set): You do not have to show that it satisfies the other vector space axioms.

(a) The set of 3-element real column vectors, with operations defined by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \oplus \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} x-u \\ y-v \\ z-w \end{bmatrix}$

and (for real numbers r) $r \odot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} rx \\ ry \\ rz \end{bmatrix}$.

(b) Let $\{a_n\}$ denote a sequence of real numbers, such as the sequence $a_n = \frac{1}{n}$ which could be partially written out as $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. Let V be the set of those sequences for which only finitely many terms are non-zero: For example, that $\frac{1}{n}$ sequence would not be in V, but if f is any polynomial, the sequence $a_n = f^{(n)}(1)$ consisting of the derivatives of f evaluated at 1 would be in V since after the k^{th} derivative any polynomial of degree k gives 0. Define addition of two sequences "termwise", i.e. $\{a_n\} + \{b_n\} = \{a_n + b_n\}$, and scalar multiplication similarly, $r\{a_n\} = \{r\, a_n\}$.

Problem 3 (18 points)

Each part of this problem gives you a vector space V and a set S of vectors in V. Tell whether the given set of vectors is linearly dependent or independent, and whether the given set spans the V. Be sure to give evidence justifying your answers.

(a) V is the subspace of $\mathbb{R}^3$ consisting of the vectors whose first and third entries are equal, and S is the set of three vectors $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \right\}$.

(b) V is the subspace of P_3 consisting of the polynomials $p(x)$ satisfying $p(0) = 0$. S is the set of three polynomials $\{x^2 + x, x^2 - x, x^3\}$.

(c) V is the set of 2×2 symmetric matrices, and the set S is $\left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \right\}$.

Problem 4 (16 points)
Problem 3 on our second midterm said:

Find a basis for the subspace W of $\mathbb{R}^3$ spanned by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix} \right\}$.

It turned out that a basis consisted of two vectors.

(a) Could there be a basis for W that (i) consisted of some the vectors $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 11 \\ 10 \\ 7 \end{bmatrix}$, and $\begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix}$, and (ii) was orthogonal (using the standard dot product on $\mathbb{R}^3$ as the inner product)? Why or why not? You do not need to find such a basis, if it exists, but you must give reasons for your answer.

(b) Could there be an orthogonal basis for W if we did not require the members of the basis to be some of those four vectors? Why or why not? You do not need to find such a basis, if it exists, but you must give reasons for your answer.

Problem 5 (18 points)

Let V be the set of continuous functions defined on $[-1, 1]$ and taking on real values. Define a function L on V by $L(f) = \int_{-1}^{1} f(t)\,dt$, for every $f \in V$.

(a) Show that L is a linear transformation from V to $\mathbb{R}$.

(b) What is the kernel of L? Give some examples of functions in the kernel, and describe in terms of the graph of f what functions f are in the kernel.

(c) What is the range of L? Tell which members of $\mathbb{R}$ are in the range.

Problem 6 (15 points)

Let S be the basis for $\mathbb{R}^3$ consisting of $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Let T be the basis for $\mathbb{R}^2$ consisting of $\vec{w}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Suppose L is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ such that the matrix representing L with respect to S and T is $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

(a) Compute $[L(\vec{v}_1)]_T$, $[L(\vec{v}_2)]_T$, and $[L(\vec{v}_3)]_T$.

(b) Compute $L(\vec{v}_1)$, $L(\vec{v}_2)$, and $L(\vec{v}_3)$.

(c) Compute $L\left(\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right)$.

Problem 7 (16 points)

We defined similarity of $n \times n$ matrices by: An $n \times n$ matrix B is similar to an $n \times n$ matrix A if there is some nonsingular matrix P such that $B = P^{-1}AP$.

Prove that similarity is an equivalence relation, i.e.: (i) for any $n \times n$ matrix A, A is similar to A; (ii) if B is similar to A, then A is similar to B; (iii) if B is similar to A and C is similar to B, then C is similar to A.

<u>Problem 8</u> (18 points)

Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 1 & 2 & -2 \end{bmatrix}$.

(a) Compute the adjoint $adj(A)$.

(b) Compute the product $adj(A)$ times A.

(c) Use your answer to (b) to find the determinant of A.

Problem 9 (17 points)

Let $A = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. The eigenvalues of A are 1 and 2.

Show that A is diagonalizable, and find a matrix P such that $P^{-1}AP$ is diagonal.

Problem 10 (17 points)

Let $A = \begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix}$.

(a) What is the characteristic polynomial of A?

(b) Find the eigenvalues of A.

(c) For each eigenvalue find the eigenvectors of A.

Problem 11 (16 points)
Let V be the real vector space of all polynomials with real coefficients (not restricted as to degree). Define a function on pairs of polynomials by

$$(p(t), q(t)) = \int_0^1 p(t)\, q(t)\, dt \qquad \text{for polynomials } p \text{ and } q.$$

This does give an inner product on V.

(a) What is the magnitude $\|p(t)\|$ of the vector (polynomial) $p(t) = t^2 - 1$?

(b) What is $\cos\theta$ if θ is the angle between $p(t) = t - 1$ and $q(t) = t + 1$?

Problem 12 (18 points)

Let W be the subspace of $\mathbb{R}^3$ spanned by $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$.

(a) Use the Gram-Schmidt process to find an orthogonal basis for W.

(b) Find an orthonormal basis for W.

Scratch Paper

(not a command!)