Your Name: ______________________________

Mathematics 340, Spring 2007 **Lecture 1 (Wilson)**

First Midterm Exam February 22, 2007

Write your answers to the nine problems in the spaces provided. If you must continue an answer somewhere other than immediately after the problem statement, be sure (a) to tell where to look for the answer, and (b) to label the answer wherever it winds up. In any case, be sure to make clear what is your final answer to each problem.

There is a problem on the back of this sheet: Be sure not to skip over it by accident!

There is scratch paper at the end of this exam. If you need more scratch paper, please ask for it.

You may refer to notes you have brought on an index card, as announced in class and on the class website.

BE SURE TO SHOW YOUR WORK, AND EXPLAIN WHAT YOU DID. YOU MAY RECEIVE REDUCED OR ZERO CREDIT FOR UNSUBSTANTIATED ANSWERS. If you are citing a theorem or other result from the book, you do not need to give theorem number or page, but you should make clear exactly what the cited result says. Don't invent new things in the book that make a problem trivial. And if the problem asks you to prove something that is in the book, don't just cite that proof!

The symbol $\mathbb{R}$ is used to denote the set of all real numbers.

Problem	Points	Score
1	12	
2	10	
3	10	
4	10	
5	12	
6	12	
7	10	
8	12	
9	12	
TOTAL	100	

Problem 1 (12 points)
Consider the system of equations:

$$\begin{aligned} 2x_1 + 3x_2 - x_3 + x_4 &= -6 \\ x_1 - x_2 + x_3 - x_4 &= 6 \\ x_2 + x_3 + x_4 &= -4 \\ x_1 - x_2 + x_3 + x_4 &= 4 \end{aligned}$$

(a) Write this in matrix form $A\vec{x} = \vec{b}$, i.e. show what A and $\vec{b}$ are for this form.

(b) Write the augmented matrix $[A \mid \vec{b}]$.

(c) Use row reduction to find all solutions of this system. (There is at least one solution. If your work leads you to believe otherwise, check your arithmetic!)

(d) Plug one of your solution(s) (there might be only one) into one of the original equations (your choice) to show that your solution fits that equation.

Problem 2 (10 points)

(a) Give a (non-zero) example of a 3×3 skew-symmetric matrix.

(b) Is it possible to find a 3×3 matrix that is: (i) not zero, (ii) a scalar matrix, and (iii) skew-symmetric? Explain your reasoning.

Problem 3 (10 points)
For $A\vec{x} = \vec{0}$ to have any non-trivial solutions, we know A must be singular. Find a scalar r such that

$$\begin{bmatrix} 1 & r \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

has some non-trivial solutions, and find one of those solutions.

Problem 4 (10 points)
Prove: If A is a nonsingular $n \times n$ matrix, then
For any $n \times 1$ vector $\vec{b}$, the system $A\vec{x} = \vec{b}$ has one and only one solution vector $\vec{x}$.
(Hint: Use A^{-1}.)

Problem 5 (12 points)

For $A = \begin{bmatrix} 2 & 0 & 4 & 6 \\ 1 & 1 & -1 & 3 \\ 2 & 2 & -2 & 4 \end{bmatrix}$:

(a) Find a matrix C in Reduced Row Echelon form such that C is row equivalent to A.

(b) (for the same A) Find all solutions to $A\vec{x} = \vec{0}$.

(c) (for the same A) For $\vec{b} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, find all solutions to $A\vec{x} = \vec{b}$.

Problem 6 (12 points)

(a) Prove: Any elementary matrix is nonsingular.

(b) Prove: If B is row equivalent to A, then $B = PA$ for some nonsingular matrix P.

Problem 7 (10 points)

$$\text{For } A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 11 \\ 1 & -3 & 9 \end{bmatrix} :$$

Is A singular, or is A nonsingular?
Give reasons for your answer, and if A is nonsingular find A^{-1}.

Problem 8 (12 points)
For any vector space V, and any vector $\vec{u} \in V$, we let $-\vec{u}$ denote a vector such that $\vec{u} \oplus (-\vec{u}) = \vec{0}$.

(a) Prove that, for any particular vector $\vec{u}$, this vector $-\vec{u}$ is uniquely determined. I.e., if $\vec{v}$ is any vector that added to $\vec{u}$ gives $\vec{0}$, then $\vec{v}$ must be the same as $-\vec{u}$. (Hint: If $\vec{v}$ is such a vector, compute $\vec{v} \oplus \vec{u} \oplus (-\vec{u})$ in two ways.)

(b) Prove: $-(-\vec{u}) = \vec{u}$.

Problem 9 (12 points)

For each of the following sets with the given operations, tell whether it is or is not a vector space with real scalars. Justify your answers. (If you think it is a vector space you do not have to prove carefully that it satisfies all eight of the properties in the definition. Do indicate how you know it is closed under $\oplus$ and $\odot$, what element in the set plays the role of $\vec{0}$, and for each $\vec{u}$ what element plays the role of $-\vec{u}$.)

(a) $V =$ the set of all quadratic polynomials with zero as the coefficient on x, i.e. $V = \{ax^2 + bx + c \mid a \in \mathbb{R} \text{ and } c \in \mathbb{R} \text{ and } b = 0\}$, with the usual way of adding polynomials as $\oplus$ and the usual way of multiplying by a scalar as $\odot$.

(b) $V =$ the set of all quadratic polynomials with a non-zero constant term, i.e. $V = \{ax^2 + bx + c \mid a \in \mathbb{R} \text{ and } b \in \mathbb{R} \text{ and } c \in \mathbb{R} \text{ and } c \neq 0\}$, with the usual way of adding polynomials as $\oplus$ and the usual way of multiplying by a scalar as $\odot$.

(c) $V =$ the set of all ordered pairs of real numbers, i.e. $V = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$, with $(x, y) \oplus (u, v) = (x + u, y + v)$ and $c \odot (x, y) = (0, 0)$.

(d) $V = M_{m,n}$ for some positive whole numbers m and n, i.e. the set of all matrices with real entries, of size $m \times n$, with $\oplus$ and $\odot$ the usual operations of addition and scalar multiplication for matrices.

Scratch Paper

(not a command!)