Mathematics 340, Spring 2007 **Lecture 1 (Wilson)**
Final Exam **May 14, 2007** **ANSWERS**

Problem 1 (15 points)

(a) There are three kinds of elementary row operations and associated elementary matrices. Describe what each kind of operation does when applied to a matrix.

ANSWER: The three types of elementary row operations (i) interchange two rows, (ii) multiply a row by some non-zero real number, and (iii) add a multiple of one row to a different row.

(b) Why is every elementary matrix non-singular? Give a proof. Your proof should make use of part (a) and should not make use of the determinant.

ANSWER: An elementary matrix is the result of applying an elementary row operation to an appropriately sized identity matrix, and multiplication by that matrix effects the corresponding row operation. For each of the three types of elementary row operations: (i) interchanging the same two rows puts the matrix back as it was, (ii) multiplication by 1/(the non-zero real number) on the same row puts the matrix back as it was, and (iii) subtracting the same multiple of the same row from the same different row will put the matrix back as it was. So each elementary row operation can be "undone" by another elementary row operation.

For any elementary matrix, consider the row operation it carries out. Now look at the elementary row operation that undoes that. Finally, take the elementary matrix that corresponds to the "undoing" operation. That elementary matrix must be the inverse of the given elementary matrix so that is non-singular.

(c) Prove that every non-singular matrix can be written as a product of elementary matrices. You may assume that the inverse of an $n \times n$ matrix A can be found, if it exists, by row-reducing an $n \times 2n$ matrix composed of A and I_n side-by-side.

ANSWER: If we write the matrix $[A|I_n]$ and row reduce it we will get $\left[I_n|A^{-1}\right]$. Since the row reduction was accomplished by a sequence of elementary row operations, and each row operation is effected by multiplying by an elementary matrix, we have that $A^{-1} = E_k E_{k-1} \cdots E_2 E_1 \, I_n$ for some elementary matrices $E_1, \ldots, E_k$, I_n is the elementary matrix corresponding to multiplying the first row by 1. Hence A^{-1} is a product of elementary matrices. But as noted in (b) the inverse of each elementary matrix is itself an elementary matrix, so $A = \left(A^{-1}\right)^{-1} = I_n E_1^{-1} E_2^{-1} \cdots E_k^{-1}$ is a product of elementary matrices.

Problem 2 (16 points)

For each of the following sets and operations, tell whether it is or is not a real vector space. If it is not, show a property a vector space must have that this one fails. (Be specific and make clear why this example fails to have that property.) f it is a vector space, show how you know it has the closure properties for addition and scalar multiplication (i.e., that the sum and scalar product defined for it always produce results within the set): You do not have to show that it satisfies the other vector space axioms.

(a) The set of 3-element real column vectors, with operations defined by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \oplus \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} x-u \\ y-v \\ z-w \end{bmatrix}$

and (for real numbers r) $r \odot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} rx \\ ry \\ rz \end{bmatrix}$.

ANSWER: This is not a vector space. As one example of how it fails: The column $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is the only thing that works as a zero vector on the right, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \oplus \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, but it does not work on the left. Or you could point out that addition is not commutative.

(b) Let $\{a_n\}$ denote a sequence of real numbers, such as the sequence $a_n = \frac{1}{n}$ which could be partially written out as $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. Let V be the set of those sequences for which only finitely many terms are non-zero: For example, that $\frac{1}{n}$ sequence would not be in V, but if f is any polynomial, the sequence $a_n = f^{(n)}(1)$ consisting of the derivatives of f evaluated at 1 would be in V since after the k^{th} derivative any polynomial of degree k gives 0. Define addition of two sequences "termwise", i.e. $\{a_n\} + \{b_n\} = \{a_n + b_n\}$, and scalar multiplication similarly, $r\{a_n\} = \{r\, a_n\}$.

ANSWER: This is a vector space. We need to show it is closed under addition and scalar multiplication. I.e., if we add two sequences each of which has only finitely many terms $\neq 0$ do we get a sequence that has only finitely many terms $\neq 0$, and similarly for $r\{a_n\}$. Given $\{a_n\}$ and $\{b_n\}$ in V, there is some number N_1 such that whenever $n > N_1$, $a_n = 0$, and there is some number N_2 such that whenever $n > N_2$, $b_n = 0$. Let N be the larger of N_1 and N_2. Then whenever $n > N$, $a_n + b_n = 0 + 0 = 0$, so $\{a_n\} + \{b_n\}$ is in V. And whenever $n > N_1$, $r\, a_n = r \times 0 = 0$, so $r\{a_n\}$ is in V.

Problem 3 (18 points)

Each part of this problem gives you a vector space V and a set S of vectors in V. Tell whether the given set of vectors is linearly dependent or independent, and whether the given set spans the V. Be sure to give evidence justifying your answers.

(a) V is the subspace of $\mathbb{R}^3$ consisting of the vectors whose first and third entries are equal, and S is the set of three vectors $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \right\}$.

ANSWER: These span V but they are not linearly independent, they are linearly dependent. Note that by subtracting the first from the second you can get $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and that by subtracting twice that from the first and then multiplying by $\frac{1}{2}$ you can get $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and that any vector whose first and third entries must be of the form $\begin{bmatrix} x \\ y \\ x \end{bmatrix}$ which is $x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. So any vector in V is a linear combination of linear combinations of these vectors and so is a linear combination of these vectors, i.e. the vectors span V. There are many different ways to show they are linearly dependent. For example, you could solve and find that the linear combination $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Or you could observe that the two vectors found above, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, are a basis for V and so no linearly independent set can have more than two vectors.

(b) V is the subspace of P_3 consisting of the polynomials $p(x)$ satisfying $p(0) = 0$. S is the set of three polynomials $\{x^2 + x, x^2 - x, x^3\}$.

ANSWER: These three polynomials are linearly independent and they span V. Any polynomial in V is of the form $a\,x^3 + b\,x^2 + c\,x + 0$. A combination of the three polynomials in S, say $r(x^2 + x) + s(x^2 - x) + t\,x^3 = t\,x^3 + (r+s)x^2 + (r-s)x$, gives $a\,x^3 + b\,x^2 + c\,x$ if and only if $t = a$, $r + s = b$, and $r - s = c$, which have unique solutions for any values of a, b, and c. Hence the polynomial $a\,x^3 + b\,x^2 + c\,x$ can be written as a linear combination of the members of S in one and only one way, i.e. the members of S are a basis for V.

(c) V is the set of 2×2 symmetric matrices, and the set S is $\left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \right\}$.

ANSWER: These cannot span V since V has dimension four and there are only two vectors. But they are linearly independent: If $a \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + b \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $a + b = 0$ and $2a + 3b = 0$ from which you can get that $a = b = 0$.

Problem 4 (16 points)
Problem 3 on our second midterm said:

Find a basis for the subspace W of $\mathbb{R}^3$ spanned by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix} \right\}$.

It turned out that a basis consisted of two vectors.

(a) Could there be a basis for W that (i) consisted of some the vectors $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 7 \end{bmatrix}$, and $\begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix}$, and (ii) was orthogonal (using the standard dot product on $\mathbb{R}^3$ as the inner product)? Why or why not? You do not need to find such a basis, if it exists, but you must give reasons for your answer.

ANSWER: No: Any basis for W would have to consist of two vectors, since one basis did. But no two vectors from these four have dot product equal to zero, so no two of these are orthogonal.

(b) Could there be an orthogonal basis for W if we did not require the members of the basis to be some of those four vectors? Why or why not? You do not need to find such a basis, if it exists, but you must give reasons for your answer.

ANSWER: Yes: Pick some pair from these four vectors that are linearly independent, e.g. the first two, and apply Gram-Schmidt.

Problem 5 (18 points)
Let V be the set of continuous functions defined on $[-1, 1]$ and taking on real values. Define a function L on V by $L(f) = \int_{-1}^{1} f(t)\,dt$, for every $f \in V$.

(a) Show that L is a linear transformation from V to $\mathbb{R}$.

ANSWER: The function L certainly takes members of V and produces numbers since that definite integral will exist for a continuous function. So we just have to show $L(f+g) = L(f) + L(g)$ and $L(rf) = rL(f)$ for any functions f and g in V and any number r. Using properties of integrals, $L(f+g) = \int_{-1}^{1}(f(t)+g(t))dt = \int_{-1}^{1} f(t)dt + \int_{-1}^{1} g(t)dt = L(f) + L(g)$ and $L(rf) = \int_{-1}^{1} r\,f(t)dt = r\int_{-1}^{1} f(t)dt = rL(f)$.

(b) What is the kernel of L? Give some examples of functions in the kernel, and describe in terms of the graph of f what functions f are in the kernel.

ANSWER: The kernel of L will be the functions f such that $\int_{-1}^{1} f(t)dt = 0$. One such function is $f(x) = x$. Another is $f(x) = \sin(\pi x)$. In general a continuous function f will be in the kernel if and only if its graph on $[-1, 1]$ encloses as much area above the x-axis as below.

(c) What is the range of L? Tell which members of $\mathbb{R}$ are in the range.

ANSWER: The range is the set of all real numbers $\mathbb{R}$. Thinking in terms of area, as in part (b), for any number r we can connect the point $(-1, 0)$ to the point $(1, r)$ by a straight line and form a triangle with base 2 and height either r (if r is positive) or $-r$, so the area (with appropriate $\pm$ sign attached) and hence the integral will be $\frac{1}{2} \times 2 \times r = r$, where the function f has that straight line for its graph. (If you don't want to think in pictures, let $f(t) = \frac{r}{2}(x+1)$ and compute the integral.)

Problem 6 (15 points)
Let S be the basis for $\mathbb{R}^3$ consisting of $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.
Let T be the basis for $\mathbb{R}^2$ consisting of $\vec{w}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
Suppose L is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ such that the matrix representing L with respect to S and T is $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.
(Comment, not printed on the exam: This is problem 22 from page 323 in the textbook, one of the problems assigned on the first day of class...)

(a) Compute $[L(\vec{v}_1)]_T$, $[L(\vec{v}_2)]_T$, and $[L(\vec{v}_3)]_T$.

ANSWER: If we were computing the matrix representing L we would fill in as the first column $[L(\vec{v}_1)]_T$, the second column $[L(\vec{v}_2)]_T$, and the third column $[L(\vec{v}_3)]_T$. Hence the answers are just the columns of the matrix, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(b) Compute $L(\vec{v}_1)$, $L(\vec{v}_2)$, and $L(\vec{v}_3)$.

ANSWER: Since we know their coordinates with respect to T, from (a), we can find the vectors themselves as linear combinations of the vectors $\vec{w}_1$ and $\vec{w}_2$ making up T: $L(\vec{v}_1) = 1\vec{w}_1 - 1\vec{w}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. $L(\vec{v}_2) = 2\vec{w}_1 + 1\vec{w}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. $L(\vec{v}_3) = 1\vec{w}_1 + 0\vec{w}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) Compute $L\left(\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right)$.

ANSWER: First we find coordinates with respect to S of this vector: If $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$ we have $2 = -a + c$, $1 = a + b$, and $-1 = b$. Then $a = 2$ and $c = 4$, so we have the coordinates as $\left[\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right]_S = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ Now we can multiply by the given matrix, $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ $= \begin{bmatrix} 4 \\ -3 \end{bmatrix}$, to get the coordinates of the vector we want, with respect to T. So the answer is $4\vec{w}_1 - 3\vec{w}_2 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$.

Problem 7 (16 points)
We defined similarity of $n \times n$ matrices by: An $n \times n$ matrix B is similar to an $n \times n$ matrix A if there is some nonsingular matrix P such that $B = P^{-1}AP$.
Prove that similarity is an equivalence relation, i.e.: (i) for any $n \times n$ matrix A, A is similar to A; (ii) if B is similar to A, then A is similar to B; (iii) if B is similar to A and C is similar to B, then C is similar to A.
ANSWER:
For (i): Let $P = I_n$, so P^{-1} is also I_n. Then $A = I_n A I_n = P^{-1}AP$ so A is similar to A.
For (ii): If B is similar to A, there is some matrix P such that $B = P^{-1}AP$. Let $Q = P^{-1}$, so $Q^{-1} = P$. Then multiplying on the right by P^{-1} and on the left by P we get $A = PBP^{-1} = Q^{-1}BQ$. So by our definition A is similar to B.
For (iii): If B is similar to A, $B = P^{-1}AP$ for some P. If C is similar to B, $C = Q^{-1}BQ$ for some Q. Then $C = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ)$ and we are through.

Problem 8 (18 points)
Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 1 & 2 & -2 \end{bmatrix}$.

(a) Compute the adjoint $adj(A)$.

ANSWER: The adjoint is the transpose of the matrix of cofactors. It will be a 3×3 matrix which has in row i and column j $(-1)^{i+j}$ times the determinant of the matrix we would get by deleting the j^{th} row and i^{th} column of A. In the upper left position where $i = j = 1$, for example, we have $(-1)^2 \det \begin{bmatrix} -1 & 2 \\ 2 & -2 \end{bmatrix} = 2 - 4 = -2$. In the position where $i = 2$ and $j = 3$ we have $(-1)^5 \det \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = (-1)(2 - 0) = -2$. Continuing in this fashion we get $\begin{bmatrix} -2 & 0 & 1 \\ 2 & -1 & -2 \\ 1 & -1 & -1 \end{bmatrix}$.

(b) Compute the product $adj(A)$ times A.

ANSWER:

(During the exam several people asked if that should be $A \times adj(A)$, i.e. put the matrices in the opposite order. Since you have in front of you two 3×3 matrices you could multiply them in either order, so I was not quite sure why the question arose. My suggestion was to try both ways and compare: Since the product is either the zero matrix (if A were singular) or $det(A)I_3$, i.e. $adj(A)$ is a number times A^{-1}, and any matrix commutes with its inverse, you are bound to get the same answer in either order.)

We know we should get a diagonal matrix, so this gives a check of our work in (a). Multiplying either $adj(A)\,A$ or $A\,adj(A)$ we get $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(c) Use your answer to (b) to find the determinant of A.

ANSWER: We really know more, the product should be the determinant of A times the identity matrix, so the determinant is the -1 we found on the diagonal.

Problem 9 (17 points)

Let $A = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. The eigenvalues of A are 1 and 2.

Show that A is diagonalizable, and find a matrix P such that $P^{-1}AP$ is diagonal.

ANSWER: We find eigenvectors associated with the given eigenvalues.

For $\lambda = 1$ we need the non-zero solutions of $\begin{bmatrix} 1 & 2 & -1 \\ -1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Row reducing, or solving any other way, we get the solutions to be all multiples of $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ plus all multiples of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Hence $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are two linearly independent eigenvectors going with $\lambda = 1$.

For $\lambda = 2$ we need the non-zero solutions of $\begin{bmatrix} 2 & 2 & -1 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The solutions are all multiples of $\vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, so the eigenvectors are the non-zero multiples of $\vec{v}_3$.

Now we know that eigenvectors going with distinct eigenvalues are linearly independent, and A is 3×3, so we know that A is diagonalizable and we can make a matrix P that diagonalizes A by using as the columns of P three linearly independent eigenvectors. Hence one choice for P is $\begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Problem 10 (17 points)

Let $A = \begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix}$.

(a) What is the characteristic polynomial of A?

ANSWER: The characteristic polynomial will be the determinant of $\begin{bmatrix} \lambda - 4 & -3 \\ -2 & \lambda - 3 \end{bmatrix}$ which is $(\lambda - 4)(\lambda - 3) - 6 = \lambda^2 - 7\lambda + 12 - 6 = \lambda^2 - 7\lambda + 6$.

(b) Find the eigenvalues of A.

ANSWER: You could use the quadratic formula, but the characteristic polynomial factors nicely as $(\lambda - 1)(\lambda - 6)$ so the roots and hence the eigenvalues must be $\lambda = 1$ and $\lambda = 6$.

(c) For each eigenvalue find the eigenvectors of A.

ANSWER: For $\lambda = 1$ we solve $\begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and get for the eigenvectorsthe nonzero multiples of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. For $\lambda = 6$ we solve $\begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and get for the eigenvectors the nonzero multiples of $\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$.

<u>Problem 11</u> (16 points)

Let V be the real vector space of all polynomials with real coefficients (not restricted as to degree). Define a function on pairs of polynomials by

$$(p(t), q(t)) = \int_0^1 p(t)\, q(t)\, dt \qquad \text{for polynomials } p \text{ and } q.$$

This <u>does</u> give an inner product on V.

(a) What is the magnitude $\|p(t)\|$ of the vector (polynomial) $p(t) = t^2 - 1$?

ANSWER: The square of the magnitude $(\|p(t)\|)^2$ will be $\int_0^1 (t^2-1)(t^2-1)dt = \int_0^1 (t^4-2t^2+1)dt$ $= \left[\frac{t^5}{5} - 2\frac{t^3}{3} + t\right]_0^1 = \frac{1}{5} - \frac{2}{3} + 1 = \frac{8}{15}$. So $\|p(t)\| = \sqrt{\frac{8}{15}}$.

(b) What is $\cos\theta$ if θ is the angle between $p(t) = t - 1$ and $q(t) = t + 1$?

ANSWER: $\cos\theta = \frac{(p(t),q(t))}{\|p(t)\|\,\|q(t)\|}$.

$(p(t), q(t)) = \int_0^1 (t-1)(t+1)dt = \int_0^1 (t^2 - 1)dt = \left[\frac{t^3}{3} - t\right]_0^1 = -\frac{2}{3}$.

$\|p(t)\|^2 = \int_0^1 (t^2 - 2t + 1)dt = \frac{1}{3} - 1 + 1 = \frac{1}{3}$. Hence $\|p(t)\| = \sqrt{\frac{1}{3}}$.

$\|q(t)\|^2 = \int_0^1 (t^2 + 2t + 1)dt = \frac{7}{3}$. Hence $\|q(t)\| = \sqrt{\frac{7}{3}}$.

Thus $\cos\theta = \frac{-\frac{2}{3}}{\sqrt{\frac{1}{3}}\sqrt{\frac{7}{3}}} = -\frac{2\sqrt{7}}{7}$.

<u>Problem 12</u> (18 points)

Let W be the subspace of $\mathbb{R}^3$ spanned by $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$.

(a) Use the Gram-Schmidt process to find an orthogonal basis for W.

ANSWER: If we call those vectors $\vec{u}_1$ and $\vec{u}_2$, we use for our basis $\vec{v}_1 = \vec{u}_1$ and $\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}$. That makes $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} - \frac{-2+0+3}{1+0+1} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$, so our orthogonal basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{5}{2} \\ 1 \\ \frac{5}{2} \end{bmatrix} \right\}$. (Note those are indeed orthogonal, their dot product is zero.)

(b) Find an orthonormal basis for W.

ANSWER: We can use those two vectors after correcting their magnitudes. Using $\frac{1}{\|\vec{v}_i\|}\vec{v}_i$ instead of $\vec{v}_i$, for $i = 1$ and 2, we have $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{\sqrt{2}}{\sqrt{27}} \begin{bmatrix} -\frac{5}{2} \\ 1 \\ \frac{5}{2} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{5}{\sqrt{54}} \\ \frac{\sqrt{2}}{\sqrt{27}} \\ \frac{5}{\sqrt{54}} \end{bmatrix} \right\}$.