Mathematics 340, Spring 2007 — Lecture 1 (Wilson)

Second Midterm Exam — March 29, 2007 — Answers

Problem 1 (13 points)

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(a) Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

ANSWER:

We need somehow to show that this set of three vectors spans $\mathbb{R}^3$ and is linearly independent. We can do some calculations or we can make use of some theorems to shorten the calculations. For example, since $\mathbb{R}^3$ has dimension 3 and there are 3 vectors in the set, we can prove either linear independence or spanning and the other will also have to be true. But, looking ahead at part (b), we are going to need to be able to find numbers x, y, and z making $x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3$ give a particular vector: If we do matrix arithmetic to find such numbers that would work for any given vector in $\mathbb{R}^3$ that would prove the vectors span, so we can do both parts at once. We row reduce the augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 2 & 1 & b \\ 1 & 3 & 0 & c \end{array}\right]$ and get $\left[\begin{array}{ccc|r} 1 & 0 & 0 & \frac{1}{2}(3a-c) \\ 0 & 1 & 0 & \frac{1}{2}(c-a) \\ 0 & 0 & 1 & \frac{1}{2}(2b-a-c) \end{array}\right]$. We can now see that we could solve the system, no matter what values a, b, and c took, so the three vectors do span $\mathbb{R}^3$ and we are through with part (a).

(b) Express $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

ANSWER:

Continuing from where we got to in (a): To represent $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ we choose $a = 2$, $b = 1$, and $c = 4$. From the RREF above we can read off a solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ as $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. So $1 \times \vec{v}_1 + 1 \times \vec{v}_2 - 2 \times \vec{v}_3$ should give $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ which is easily checked to be true.

Problem 2 (13 points)

Show that if $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and $\vec{v}_3$ does not belong to the span of $\{\vec{v}_1, \vec{v}_2\}$ then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

ANSWER:
First note that $\vec{v}_3 \neq \vec{0}$ since the zero vector is in the span of any collection of vectors. Also, $\vec{v}_1 \neq \vec{0}$ and $\vec{v}_2 \neq \vec{0}$ since the set $\{\vec{v}_1, \vec{v}_2\}$ was given to be linearly independent, while no set containing $\vec{0}$ is linearly independent. Thus each of the vectors in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is different from $\vec{0}$. A theorem we had in the book says that such a set is linearly dependent only if at least one vector is a linear combination of the vectors that come before it. $\vec{v}_1$ is not a linear combination of the (empty set) of vectors before it, since it is not $\vec{0}$. $\vec{v}_2$ is not a linear combination of the vectors that came before it, just $\vec{v}_1$, since $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. And $\vec{v}_3$ is given not to be a linear combination of the vectors ($\vec{v}_1$ and $\vec{v}_2$) that come before it. So no vector in the set is a linear combination of the preceding vectors, so the set is not linearly dependent, so it is linearly independent.

Problem 3 (12 points)

Find a basis for the subspace W of $\mathbb{R}^3$ spanned by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix} \right\}$.
What is the dimension of W?

Hint: The RREF of $\begin{bmatrix} 1 & 3 & 11 & 7 \\ 2 & 2 & 10 & 6 \\ 2 & 1 & 7 & 4 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

ANSWER:
The space W spanned by those four vectors is by definition the column space of the matrix $\begin{bmatrix} 1 & 3 & 11 & 7 \\ 2 & 2 & 10 & 6 \\ 2 & 1 & 7 & 4 \end{bmatrix}$. We can use as a basis for the column space the vectors (columns) in that matrix that match leading entries in the RREF matrix, i.e. the first and second columns. So one basis for the subspace W is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$.
Since that basis has two vectors, the dimension of W is two.

Problem 4 (12 points)

(a) Find a basis for, and the dimension of, the solution space of the system

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 &= 0 \\ 2x_1 + 2x_2 - x_3 + 2x_4 &= 0 \\ x_1 \qquad + 3x_3 + 3x_4 &= 0 \end{aligned}$$

Hint: The RREF of $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 2 & -1 & 2 \\ 1 & 0 & 3 & 3 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{8}{3} \\ 0 & 0 & 1 & \frac{4}{3} \end{bmatrix}$.

ANSWER:

Let $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 2 & -1 & 2 \\ 1 & 0 & 3 & 3 \end{bmatrix}$. From the RREF we can see that the solutions to $A\vec{x} = \vec{0}$ have the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where x_4 can be arbitrary, $x_1 = x_4$, $x_2 = -\frac{8}{3}x_4$, and $x_3 = -\frac{4}{3}x_4$. Hence all solutions are multiples of $\begin{bmatrix} 1 \\ -\frac{8}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$, i.e. the set with just one vector $\left\{ \begin{bmatrix} 1 \\ -\frac{8}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix} \right\}$ is a basis for the solution space, and the dimension is 1.

(b) Let $A = \begin{bmatrix} 1 & -2 & 7 & 0 \\ 1 & -1 & 4 & 0 \\ 3 & 2 & -3 & 5 \\ 2 & 1 & -1 & 3 \end{bmatrix}$

Find a basis for the column space of A, consisting of some columns of A.

Hint: The RREF of A is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

ANSWER:

We can use the columns of A corresponding to leading entries in the RREF matrix, i.e. the first, second, and fourth columns. Hence one basis for the column space, consisting of columns of A, is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \\ 3 \end{bmatrix} \right\}$.

Problem 5 (13 points)

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 9 & -1 \\ -3 & 8 & 3 \\ -2 & 3 & 2 \end{bmatrix}$

Find a basis for the row space of A, consisting of vectors that are rows of A.

ANSWER:

The most straightforward way to do this is probably this: First make the rows into columns, i.e. work on A^T rather than A. Now find a basis for the column space of A^T composed of some columns of A^T, using the RREF of A^T. Then write those columns back as rows. Carrying this out: $A^T = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & 9 & 8 & 3 \\ -1 & -1 & 3 & 2 \end{bmatrix}$. If you row reduce that you get $\begin{bmatrix} 1 & 0 & -5 & -3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so the first and second columns of A^T, the ones where there are leading entries in the RREF, would be a basis for the column space of A^T, so the first and second rows of A will be a basis for the row space of A. Hence we can use $\{[\,1\ \ 2\ -1], [\,1\ \ 9\ -1]\}$ as our basis.

Another way: If you notice that the first entry in any row is the negative of the last entry, you can infer the row space is at most two dimensional. But since the middle entries can be anything (you can get the row [0 1 0] from the first and second rows) the dimension is at least two, hence it is two. So any two linearly independent rows will work. The very first pair you might try, the top two rows, can be checked to be linearly independent, so they are a basis, exactly the same one we found before!

Problem 6 (12 points)

Let A be a 7×3 matrix whose rank is 3.

(a) Are the rows of A linearly independent or linearly dependent? Justify your answer.

ANSWER:

Since the rank of A is 3 we know that both the row space and the column space have dimension 3. There are 7 rows in A, which are 7 vectors in the row space of A, and a set of more vectors than the dimension cannot be linearly independent in any vector space. So they must be linearly dependent.

(b) Are the columns of A linearly independent or linearly dependent? Justify your answer.

ANSWER:

As above, the dimension of the column space of A is 3. There are three columns in A, by definition they span the column space, and there are as many of them as the dimension, so they must be a basis. Hence they must be linearly independent.

Problem 7 (12 points)

Define a function L from $\mathbb{R}^3$ to $\mathbb{R}^3$ by $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 4u_2 \\ -u_3 \\ u_2 + u_3 \end{bmatrix}$.

(a) Show that L is a linear transformation.

ANSWER:

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be any two vectors in $\mathbb{R}^3$, and c any real number.

Then $L(\vec{u} + \vec{v}) = L\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}\right)$: Applying the given rule for L, that must be

$\begin{bmatrix} (u_1+v_1)+4(u_2+v_2) \\ -(u_3+v_3) \\ (u_2+v_2)+(u_3+v_3) \end{bmatrix}$. On the other hand $L(\vec{u})+L(\vec{v}) = \begin{bmatrix} u_1+4u_2 \\ -u_3 \\ u_2+u_3 \end{bmatrix} + \begin{bmatrix} v_1+4v_2 \\ -v_3 \\ v_2+v_3 \end{bmatrix}$.
A quick calculation shows then $L(\vec{u}+\vec{v}) = L(\vec{u}) + L(\vec{v})$

Now we need to take care of scalar multiplication. $L(c\vec{u}) = L\left(\begin{bmatrix} c\,u_1 \\ c\,u_2 \\ c\,u_3 \end{bmatrix}\right) = \begin{bmatrix} c\,u_1+4c\,u_2 \\ -c\,u_3 \\ c\,u_2+c\,u_3 \end{bmatrix}$

$= c\begin{bmatrix} u_1+4u_2 \\ -u_3 \\ u_2+u_3 \end{bmatrix} = cL(\vec{u})$, and we are through.

(b) Find the standard matrix representing L, i.e. a matrix A such that $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) =$ $A\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ for any vector $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ in $\mathbb{R}^3$.

ANSWER:

We apply the transformation L to each of the three standard basis vectors for $\mathbb{R}^3$: $L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, and $L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. We assemble those as the columns of the standard matrix, $\begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

Problem 8 (13 points)
Let $L : \mathbb{R}_4 \longrightarrow \mathbb{R}_2$ be the linear transformation defined by $L([u_1\ u_2\ u_3\ u_4]) = [u_1+u_3,\ u_2+u_4]$.

(a) Is $[2\ \ 3\ \ -2\ \ 3]$ in the kernel of L?

ANSWER:

Computing, $L([2\ \ 3\ \ -2\ \ 3]) = [0\ \ 6] \neq [0\ \ 0]$, so the result is not $\vec{0}$ and $[2\ \ 3\ \ -2\ \ 3]$ is not in the kernel.

(b) Is $[4\ \ -2\ \ -4\ \ -2]$ in the kernel of L?

ANSWER:

Computing, $L([4\ -2\ -4\ -2]) = [0\ -4] \neq [0\ \ 0]$, so the result is not $\vec{0}$ and $[4\ -2\ -4\ -2]$ is not in the kernel.

(c) Find a basis for the kernel of L.

ANSWER:

The kernel of L is the set of those vectors $\vec{u}$ such that $L(\vec{u}) = \vec{0}$, i.e. $[u_1\ u_2\ u_3\ u_4]$ such that $[u_1 + u_3,\ \ u_2 + u_4] = [0\ \ 0]$. They must satisfy $u_1 + u_3 = 0$ and $u_2 + u_4 = 0$. You could use matrix methods but those equations are so simple you may be able just to read off that (i) $[1\ \ 0\ \ -1\ \ 0]$ and $[0\ \ 1\ \ 0\ \ -1]$ are solutions and (ii) any solution is a linear combination of those two vectors, so together they are a basis for the kernel of L.