First Midterm Exam February 22, 2007

ANSWERS

The symbol $\mathbb{R}$ is used to denote the set of all real numbers.

Problem 1 (12 points)
Consider the system of equations:

$$\begin{aligned} 2x_1 + 3x_2 - x_3 + x_4 &= -6 \\ x_1 - x_2 + x_3 - x_4 &= 6 \\ x_2 + x_3 + x_4 &= -4 \\ x_1 - x_2 + x_3 + x_4 &= 4 \end{aligned}$$

(a) Write this in matrix form $A\vec{x} = \vec{b}$, i.e. show what A and $\vec{b}$ are for this form.

ANSWER

A will consist of the coefficients on the unknowns and $\vec{b}$ will be the column of constants on the right side of the equations:

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} -6 \\ 6 \\ -4 \\ 4 \end{bmatrix}$$

(b) Write the augmented matrix $[A \mid \vec{b}]$.

ANSWER

$$\left[A \mid \vec{b}\right] = \left[\begin{array}{rrrr|r} 2 & 3 & -1 & 1 & -6 \\ 1 & -1 & 1 & -1 & 6 \\ 0 & 1 & 1 & 1 & -4 \\ 1 & -1 & 1 & 1 & 4 \end{array}\right]$$

(c) Use row reduction to find all solutions of this system. (There is at least one solution. If your work leads you to believe otherwise, check your arithmetic!)

ANSWER

There are many sequences of elementary row operations you could choose to apply to the augmented matrix. Here is a quick trip through one choice. First swap the first and second rows. Now swap the second and third rows. Then subtract twice the first row from the third. Subtract the first row from the fourth. At this point you should have

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 6 \\ 0 & 1 & 1 & 1 & -4 \\ 0 & 5 & -3 & 3 & -18 \\ 0 & 0 & 0 & 2 & -2 \end{array}\right]$$

Now add 5 times the first row to the third. Then multiply the fourth row by $\frac{1}{2}$. Add twice the last row to the third. Now you should have

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 6 \\ 0 & 1 & 1 & 1 & -4 \\ 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

We are almost there... Divide the third row by -8. Now all the leading entries are in place, with zeros to their left, so we have it in REF. Now we need to clear out the entries above the leading entries. Add the second row to the first, then subtract the last row from the second. This gets us to the RREF form of the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

From this we can read off that there is only one solution. In this form the first equation amounts to $x_1 = 2$, the second to $x_2 = -3$, the third to $x_3 = 0$, and the last to $x_4 = -1$.

(d) Plug one of your solution(s) (there might be only one) into one of the original equations (your choice) to show that your solution fits that equation.

ANSWER

If we use the first equation and substitute these values for the x_i variables, we have $2 \times 2 + 3 \times (-3) - 0 + (-1)$ on the left, which works out to -6, so the equation is satisfied!

Problem 2 (10 points)

(a) Give a (non-zero) example of a 3×3 skew-symmetric matrix.

ANSWER

There are infinitely many correct answers. One is $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$.

(b) Is it possible to find a 3×3 matrix that is: (i) not zero, (ii) a scalar matrix, and (iii) skew-symmetric? Explain your reasoning.

ANSWER

No, it is not possible. Suppose a matrix satisfies (ii) and (iii). Because it is a scalar matrix, the only possible non-zero entries are on the diagonal. But the diagonal entries of a skew symmetric matrix must be zero. Hence the matrix is all zeros, so it does not satisfy (i).

Problem 3 (10 points)

For $A\vec{x} = \vec{0}$ to have any non-trivial solutions, we know A must be singular. Find a scalar r such that

$$\begin{bmatrix} 1 & r \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

has some non-trivial solutions, and find one of those solutions.

ANSWER

We start by trying to row reduce the matrix. If I subtract the first row from the second I get $\begin{bmatrix} 1 & r \\ 0 & -2-r \end{bmatrix}$. If $-2 - r \neq 0$ we see that the RREF will be I_2, or equivalently that it won't have any zero rows, so the system will have only the trivial solution. Hence we have to choose r so that $-2 - r = 0$, i.e. $r = -2$. Then the RREF will be $\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ Thus x_2 could be anything, say $x_2 = 5$, so long as $x_1 = 2x_2$. Hence $\begin{bmatrix} 10 \\ 5 \end{bmatrix}$ is a solution, when $r = -2$.

Problem 4 (10 points)
Prove: If A is a nonsingular $n \times n$ matrix, then
For any $n \times 1$ vector $\vec{b}$, the system $A\vec{x} = \vec{b}$ has one and only one solution vector $\vec{x}$.
ANSWER
Suppose A is nonsingular, i.e. there is an inverse A^{-1}. Then we can multiply both sides of $A\vec{x} = \vec{b}$ on the left by A^{-1}, getting $I_n\vec{x} = A^{-1}\vec{b}$, so $\vec{x} = A^{-1}\vec{b}$ (i) is a solution and (ii) is the only solution.

Problem 5 (12 points)

For $A = \begin{bmatrix} 2 & 0 & 4 & 6 \\ 1 & 1 & -1 & 3 \\ 2 & 2 & -2 & 4 \end{bmatrix}$:

(a) Find a matrix C in Reduced Row Echelon form such that C is row equivalent to A.

ANSWER

The RREF is $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) (for the same A) Find all solutions to $A\vec{x} = \vec{0}$.

ANSWER

From the RREF in (a) we can read off three equations: $x_1 + 2x_3 = 0$, $x_2 - 3x_3 = 0$, and $x_4 = 0$. We can choose any value for x_3 and, with those equations, that will determine a solution. So if we write α for our choice for x_3, the solution is $x_1 = -2\alpha$, $x_2 = 3\alpha$, $x_3 = \alpha$, and $x_4 = 0$. There are other ways to write this, e.g. $\begin{bmatrix} -2\alpha \\ 3\alpha \\ \alpha \\ 0 \end{bmatrix}$ or $\alpha \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$.

(c) (for the same A) For $\vec{b} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, find all solutions to $A\vec{x} = \vec{b}$.

ANSWER

We go back and row reduce the matrix $\left[A \mid \vec{b}\right]$, and get $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 10 \\ 0 & 1 & -3 & 0 & -2 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right]$. As in (b) we can give any value, call it α, to x_3, and the solutions are the quadruples $x_1 = 10 - 2\alpha$, $x_2 = 3\alpha - 2$, $x_3 = \alpha$, and $x_4 = -3$.

Problem 6 (12 points)

(a) Prove: Any elementary matrix is nonsingular.

ANSWER

Let E be an $n \times n$ elementary matrix. It is produced by applying an elementary row operation to I_n, and multiplying E on the left of any (appropriately sized) matrix A will apply that row operation to A. If the row operation producing E is a row swap, then applying that same row operation again puts a matrix back in its original state. If the row operation adds a multiple of the i^{th} row to the j^{th} $(i \neq j)$ then the row operation that subtracts that multiple of the i^{th} row from the j^{th} would undo the effect of E. If E corresponds to multiplying the i^{th} row by the non-zero scalar λ, then multiplying the i^{th} row by $\frac{1}{\lambda}$ is an elementary row operation that undoes the effect of E. So no matter what elementary row operation E performs, there is an elementary row operation that undoes it and a corresponding elementary matrix F such that $FEA = A$ for every matrix A with n rows. In particular this is true if $A = I_n$, so $FE = I_n$. But that is enough to guarantee that $F = E^{-1}$, and E is nonsingular.

(b) Prove: If B is row equivalent to A, then $B = PA$ for some nonsingular matrix P.

ANSWER

If B is row equivalent to A, there is a sequence of elementary row operations that can be applied to A to yield A. Expressed as elementary matrices, we have $B = E_k E_{k-1} \cdots E_2 E_1 A$ for some k elementary matrices $E_1, \cdots, E_k$. Since each E_i is nonsingular (by (a)), $E_k E_{k-1} \cdots E_2 E_1$ is nonsingular with inverse $(E_1)^{-1}(E_2)^{-1} \cdots (E_k)^{-1}$. Let $P = E_k \cdots E_1$. Then $B = PA$ and we have shown that P has an inverse so it is nonsingular.

Problem 7 (10 points)

For $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 11 \\ 1 & -3 & 9 \end{bmatrix}$:

Is A singular, or is A nonsingular?
Give reasons for your answer, and if A is nonsingular find A^{-1}.
ANSWER
If we constuct the 3×6 matrix that has A as its first three columns and I_3 as the last three, $\left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 2 & -5 & 11 & 0 & 1 & 0 \\ 1 & -3 & 9 & 0 & 0 & 1 \end{array}\right]$, and row reduce it, we get $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 12 & -9 & 7 \\ 0 & 1 & 0 & 7 & -6 & 5 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$. Since the first three columns are I_3, the last three must be A^{-1}, i.e. $A^{-1} = \begin{bmatrix} 12 & -9 & 7 \\ 7 & -6 & 5 \\ 1 & -1 & 1 \end{bmatrix}$.

Problem 8 (12 points)
For any vector space V, and any vector $\vec{u} \in V$, we let $-\vec{u}$ denote a vector such that $\vec{u} \oplus (-\vec{u}) = \vec{0}$.

(a) Prove that, for any particular vector $\vec{u}$, this vector $-\vec{u}$ is uniquely determined. I.e., if $\vec{v}$ is any vector that added to $\vec{u}$ gives $\vec{0}$, then $\vec{v}$ must be the same as $-\vec{u}$. (Hint: If $\vec{v}$ is such a vector, compute $\vec{v} \oplus \vec{u} \oplus (-\vec{u})$ in two ways.)

ANSWER

Note that the expression "added to $\vec{u}$" does not say on which side, but that it does not matter because vector addition is commutative (property 1).

Using the hint we compute the same thing with two different arrangements of parentheses: Since vector addition was required to be associative (property 2) we have $(\vec{v} \oplus \vec{u}) \oplus (-\vec{u}) = \vec{v} \oplus (\vec{u} \oplus (-\vec{u}))$. The left side amounts to $\vec{0} \oplus (-\vec{u}) = -\vec{u}$, and the right side is $\vec{v} \oplus \vec{0} = \vec{v}$. Hence $-\vec{u} = \vec{v}$ and we are through.

(b) Prove: $-(-\vec{u}) = \vec{u}$.

ANSWER

If we add $-\vec{u} \oplus \vec{u}$ we get $\vec{0}$. By (a), the only vector we can add to $(-\vec{u})$ and get $\vec{0}$ is the negative of $-\vec{u}$, which we have denoted $-(-\vec{u})$, and we are through.

Problem 9 (12 points)
For each of the following sets with the given operations, tell whether it is or is not a vector space with real scalars. Justify your answers. (If you think it is a vector space you do not have to prove carefully that it satisfies all eight of the properties in the definition. Do indicate how you know it is closed under $\oplus$ and $\odot$, what element in the set plays the role of $\vec{0}$, and for each $\vec{u}$ what element plays the role of $-\vec{u}$.)

(a) $V =$ the set of all quadratic polynomials with zero as the coefficient on x, i.e.
$V = \{ax^2 + bx + c \mid a \in \mathbb{R} \text{ and } c \in \mathbb{R} \text{ and } b = 0\}$, with the usual way of adding polynomials as $\oplus$ and the usual way of multiplying by a scalar as $\odot$.

ANSWER

This is a vector space. If we add two such polynomials $a_1x^2 + 0x + c_1$ and $a_2x^2 + 0x + c_2$ we get $(a_1 + a_2)x^2 + 0x + (c_1 + c_2)$ which is another such polynomial, so the set is closed under $\oplus$. If we multiply by a scalar r, $r \odot (ax^2 + 0x + c) = (ra)x^2 + 0x + (rc)$ still has a zero first-power term, so we have closure for $\odot$. The polynomial $0x^2 + 0x + 0$ works as $\vec{0}$. For a polynomial $ax^2 + 0x + c$ in the set, the polynomial $-ax^2 + 0x - c$ works as its negative and is still in the set since its x term is zero.

(In the terminology of section 3.3, not included on the test, this set is a subspace of P_2, the space of all polynomials of degree at most two. Hence just showing closure and non-empty would suffice even if I had not said you could skip showing all the properties: Exhibiting what works as $\vec{0}$ shows the set is non-empty.)

(b) $V =$ the set of all quadratic polynomials with a non-zero constant term, i.e.
$V = \{ax^2 + bx + c \mid a \in \mathbb{R} \text{ and } b \in \mathbb{R} \text{ and } c \in \mathbb{R} \text{ and } c \neq 0\}$, with the usual way of adding polynomials as $\oplus$ and the usual way of multiplying by a scalar as $\odot$.

ANSWER This set and operations are not closed under either $\oplus$ or $\odot$, so this is not a vector space. If you take any element in the set and multiply by the scalar 0 you get a polynomial whose constant term is 0, so the result is not in the set. If you want to show it is not closed under $\oplus$ you have to be a bit more careful in what example you choose, e.g. $x^2 + x + 3$ and $4x^2 - 2x - 3$ are in the set but their sum has a zero constant term and so is not in the set. You can also note that the zero vector for polynomials, $0x^2 + 0x + 0$, is not in the set, but then you should say something to eliminate the worry that something else might work as $\vec{0}$.

(c) $V =$ the set of all ordered pairs of real numbers, i.e.
$V = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$, with $(x, y) \oplus (u, v) = (x + u, y + v)$ and $c \odot (x, y) = (0, 0)$.

ANSWER

This is not a vector space. It does satisfy most of the properties. But the very last property in the definition of a vector space says $1 \odot \vec{u} = \vec{u}$: In this case $1 \odot (1, 1) = (0, 0) \neq (1, 1)$ so that property is not satisfied.

(d) $V = M_{m,n}$ for some positive whole numbers m and n, i.e. the set of all matrices with real entries, of size $m \times n$, with $\oplus$ and $\odot$ the usual operations of addition and scalar multiplication for matrices.

ANSWER

This is a vector space, in fact it is one of the examples given in the text. By the way we defined addition and scalar multiplication of matrices, if A and B are $m \times n$ matrices and r is any scalar, $A + B$ and rA are again $m \times n$ matrices so the set is closed with these operations. The vector working as $\vec{0}$ is the zero matrix, $m \times n$ with all entries 0, and the negative of a vector A is its usual matrix negative, the matrix of the same size each of whose entries is the negative of the corresponding entry in A.