Mathematics 234, Fall 2005 Lecture 2 (Wilson)

Final Exam December 20, 2005 ANSWERS

Problem 1

One of the following two vector fields is conservative and the other is not:

$$\overrightarrow{F_1}(x,y) = (3+2xy)\vec{\imath} + (x^2-3y^2)\vec{\jmath}$$

$$\overrightarrow{F_2}(x,y) = (x-y)\vec{\imath} + (x-2)\vec{\jmath}$$

(a) Which vector field is conservative? Which one is not conservative?

ANSWER: Think of $\vec{F_1}$ and $\vec{F_2}$ in the form $M(x,y)\vec{\imath} + N(x,y)\vec{\jmath}$. Then for $\vec{F_1}$ we have $\frac{\partial M}{\partial y} = 2x$ and $\frac{\partial N}{\partial x} = 2x$, and since these are equal we know $\vec{F_1}$ is conservative. In the same way for $\vec{F_2}$, $\frac{\partial M}{\partial y} = -1$ and $\frac{\partial N}{\partial x} = 1$ and these are not equal so $\vec{F_2}$ is not conservative.

(b) For the vector field $\overrightarrow{F}$ that you found to be conservative, use the fact that it is conservative to evaluate

$$\int_C \overrightarrow{F} \cdot d\vec{r}$$

where C is the curve $\vec{r}(t) = e^t \sin(t)\vec{\imath} + e^t\cos(t)\vec{\jmath}$ with $0 \le t \le \pi$.

ANSWER: We will find a potential function $f(x,y)$ for $\vec{F_1}$, then evaluate it at the ends of the curve and subtract. The ends of the curve are at $\vec{r}(0) = \vec{\jmath}$ and $\vec{r}(\pi) = -e^{\pi}\vec{\jmath}$, or as points $(0,1)$ and $(0,-e^{\pi})$.

We need $\frac{\partial f}{\partial x} = 3+2xy$, so $f(x,y) = 3x + x^2y + C(y)$ where C can vary with y but not with x. We also want $\frac{\partial f}{\partial y} = x^2 - 3y^2$ so $\frac{\partial C}{\partial y}$ must be $-3y^2$, $C(y) = -y^3 + C$ where this C is a genuine constant. Hence $f(x,y) = 3x + x^2y - y^3 + C$: In evaluating the integral we can use any value of C so we choose $C = 0$ for convenience.

Now the integral can be evaluated as $f(0,-e^{\pi}) - f(0,1)$ which gives $e^{3\pi} - (-1) = 1 + e^{3\pi}$.

Problem 2

For $f(x,y) = e^{2x-2}\cos(x - \frac{y}{2})$:

(a) Find an equation for the tangent plane to the graph of $z = f(x,y)$ at the point where $x = 1$ and $y = 2$.

ANSWER: The partial derivatives are $f_x = 2e^{2x-2}\cos(x-\frac{y}{2}) - e^{2x-2}\sin(x-\frac{y}{2})$, which evalutes to 2 at the point $(1,2)$, and $f_y = \frac{1}{2}\sin(x-\frac{y}{2})$, which evaluates to 0. The value of f at $(1,2)$ is 1. Hence the tangent plane is $z = 1 + 2(x-1) + 0(y-2)$ or $2x - z = 1$.

(b) Find an approximate value for $f(1.1, 1.9)$.
Your answer must visibly use calculus! Don't just use a calculator to work out the function at that point.

ANSWER: We use the z value on the tangent plane which we assume stays fairly close to the graph. Thus we use for our approximation $z = 1 + 2(1.1 - 1) = 1.2$.

Problem 3

Use an integral to find the volume of the region that is
(a) under the paraboloid $z = x^2 + y^2$,
(b) inside the cylinder $x^2 + y^2 = 2x$, and
(c) above the plane $z = 0$.

ANSWER: The equation $x^2 + y^2 = 2x$, on completing the square, is equivalent to $(x-1)^2 + y^2 = 1$. Hence in the x-y plane it corresponds to a circle with radius 1 and center $(1, 0)$. In space we have the vertical cylinder through that circle. So we want the volume over that circular disk and below the paraboloid. In rectangular coordinates we can set this up as

$$\int_0^2 \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} (x^2 + y^2) dy\, dx.$$

It will be much easier to evaluate this if we convert to polar coordinates. The circle is $r = 2\cos\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. The integrand $x^2 + y^2$ becomes r^2, and $dy\, dx$ is replaced by $r\, dr\, d\theta$. Thus the integral becomes

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^3\, dr\, d\theta.$$

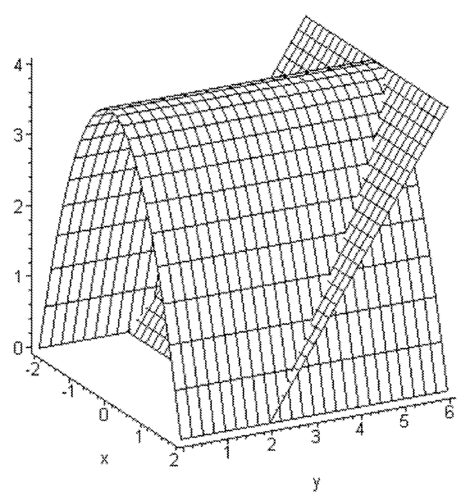
We use the identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ twice in evaluating this, and get $\frac{3}{2}\pi$.

Problem 4

Set up as an iterated integral but do not evaluate

$$\iiint\limits_S (xy - y\sin(z))\, dV$$

where S is the region in space that is
(a) under the parabolic sheet $z = 4 - x^2$,
(b) above the coordinate plane $z = 0$,
(c) where $y \ge 0$, and
(d) above the plane $z = y - 2$.



ANSWER: The integration region is shown in the picture to the right, where I have rotated the viewpoint bringing the x-axes somewhat to the right so that you can see under the "roof" and see how the plane cuts up through the space.
One way to set this up is to start with dz on the outside, i.e. start by slicing across the z-axis. The top of the roof is at $z = 4$ and the bottom of the figure is at $z = 0$. At any z value between 0 and 4, a horizontal slice finds a rectangle with one end in the x-z plane, i.e. where $y = 0$, and the other end on the plane $z = y - 2$ or $y = z + 2$. The other two sides of the rectangle, the ones parallel to the y-axis, are where $x = \pm\sqrt{4 - z}$. The fact that this is a rectangle implies that we can choose to put dx next and then dy, or the other way around, equally well. If se choose dy and then dx, working inward, we get

$$\int_0^4 \int_0^{z+2} \int_{-\sqrt{4-z}}^{\sqrt{4-z}} (xy - y\sin(z))\, dx\, dy\, dz.$$

You can also start by slicing across the x-axis. This gives a trapezoid. The top and bottom edges are parallel, at $z = 0$ and $z = 4 - x^2$. The other two edges are $y = 0$ and $y = 2 + z$. Since these latter two edges are not parallel, if you choose the order $dz\,dy\,dx$ you will have to construct two integrals: The range of z values starts at 0 for some y values, but starts at $z = y - 2$ for others. Likewise, if you put dy on the outside you will have to use two integrals.

<u>Problem 5</u>
Find all local maximum points, local minimum points, and saddle points for

$$f(x, y) = x^4 - 4xy + y^4 + 2.$$

Be sure to identify each point that you list as to maximum, minimum, or saddle point. You do not need to give the values of the function at the points.
<u>ANSWER:</u> Taking the partial derivatives, $f_x = 4x^3 - 4y$ and $f_y = 4y^3 - 4x$. There are no places where $f(x, y)$ fails to be differentiable, or any boundary points, so the only places we have to look for are those where the partials are both zero. These give us $4x^3 - 4y = 0$, or $y = x^3$, and $4y^3 - 4x = 0$, or $x = y^3$. Substituting we get $y = x^3 = (y^3)^3 = y^9$, $y^9 - y = 0$, $y(y^8 - 1) = 0$. Hence we have $y = 0$, $y = 1$, or $y = -1$, and since $x = y^3$ we get the three points $(0, 0)$, $(-1, -1)$, and $(1, 1)$.
The second partial derivatives are $f_{xx} = 12x^2$, $f_{yy} = 12y^2$, and $f_{x,y} = f_{y,x} = -4$. At $(0, 0)$ we have $D = f_{xx}f_{yy} - f_{xy}^2 = 0 \times 0 - 16 = -16 < 0$, so there is a saddle point at $(0, 0)$. At $(-1, -1)$ we have $D = 12 \times 12 - 16 = 128 > 0$, so we have either a local maximum or local minimum: Since $f_{xx} = 12 > 0$ at this point, it is a local minimum. At $(1, 1)$ the second partials are exactly the same as at $(-1, -1)$ so $(1, 1)$ also gives a local minimum.

<u>Problem 6</u>

Let $f(x, y, z) = x^2y + (x - y)\cos(\pi\, z)$.

(a) Find the gradient of f at the point $(1, 2, 1)$.
<u>ANSWER:</u> $f_x = 2xy + \cos(\pi z)$ which at $(1, 2, 1)$ gives 3. $f_y = x^2 - \cos(\pi z)$ which at $(1, 2, 1)$ gives 2. $f_z = -\pi(x - y)\sin(\pi z)$ which at $(1, 2, 1)$ gives 0. Hence the gradient is $\overrightarrow{\nabla f} = 3\vec{\imath} + 2\vec{\jmath} + 0\vec{k}$.

(b) In what direction is the directional derivative of f at $(1, 2, 1)$ the largest?
What is the derivative in that direction, at that point?
<u>ANSWER:</u> The direction making the directional derivative largest is the direction of the gradient, $3\vec{\imath} + 2\vec{\jmath} + 0\vec{k}$. You could also express it as a unit vector in the same direction, which would be $\frac{1}{\sqrt{13}}(3\vec{\imath} + 2\vec{\jmath} + 0\vec{k})$. The derivative in that direction will be the magnitude of the gradient, $\sqrt{13}$.

(c) What is the derivative of f at $(1, 2, 1)$ in the direction of the vector $\vec{v} = 2\vec{\imath} + \vec{\jmath} - 2\vec{k}$?
<u>ANSWER:</u> We find a unit vector $\vec{u}$ in the same direction as $\vec{v}$ $\frac{1}{3}\vec{v} = \frac{2}{3}\vec{\imath} + \frac{1}{3}\vec{\jmath} - \frac{2}{3}\vec{k}$. Now the derivative in the direction of $\vec{u}$ will be $\overrightarrow{\nabla f} \cdot \vec{u} = \frac{6}{3} + \frac{2}{3} = \frac{8}{3}$.

Problem 7

Use Stokes' theorem to evaluate $\oint_C \vec{F}\cdot\vec{T}\,ds$ where

(i) $\vec{F}(x,y,z) = y\vec{\imath} - x\vec{\jmath} + 3\vec{k}$.

(ii) C is the circle where the plane $2x+2y+2z=0$ intersects the sphere $x^2+y^2+z^2=9$, oriented counterclockwise as viewed from above.

ANSWER: We use $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint\limits_S \overrightarrow{curl}(\vec{F})\cdot\vec{n}\,dS,$ from Stokes' theorem,

where S is the disk whose boundary is C. To use this we need to find $\vec{n}$ and $\overrightarrow{curl}(\vec{F})$.
The disk lies in the plane $2x+2y+2z=0$. We need to find a vector for $\vec{n}$ that will (a) be perpendicular to that plane, (b) have unit magnitude, and (c) have the correct direction to go with the orientation of C. The vector $2\vec{\imath}+2\vec{\jmath}+2\vec{k}$ would be perpendicular to the plane, and so would any multiple of it such as $\vec{\imath}+\vec{\jmath}+\vec{k}$. But if we used that for $\vec{u}$ we would have $|\vec{u}|=\sqrt{3}$. So we use either $\vec{u}=\frac{1}{\sqrt{3}}(\vec{\imath}+\vec{\jmath}+\vec{k})$ or the negative of that, as needed to get the right orientation. We want C oriented counterclockwise as viewed from above. If we "walk around C" in that counterclockwise direction, if we want to keep the left hand in S (i.e. inside C rather than pointing outward) we need to be standing upright, not on our heads. So we want $\vec{u}$ pointing upward. (But that does not mean $\vec{u}=\vec{k}$! That would not be perpendicular to S.) Hence we use $\vec{u}=\frac{1}{\sqrt{3}}(\vec{\imath}+\vec{\jmath}+\vec{k})$: If we used instead $\frac{1}{\sqrt{3}}(-\vec{\imath}-\vec{\jmath}-\vec{k})$, the "$-\vec{k}$" is a giveaway that it points downward.
Finding the curl of the vector field, $\overrightarrow{curl}(\vec{F})$, is a straightforward calculation that gives $-2\vec{k}$. Evaluating the dot product of that with $\vec{u}$ gives $-\frac{2}{\sqrt{3}}$. So now we know we need to evaluate

$$\iint\limits_S -\frac{2}{\sqrt{3}}\,dS.$$

Integrating a constant over a flat region will give that constant times the area of the region. S is the circular disk cut from a sphere of radius 3 by a plane through its center, so S is a circular disk of radius 3. Hence the area of S is $\pi\,3^2 = 9\pi$. The answer is then $-\frac{2}{\sqrt{3}}\times 9\pi = -6\sqrt{3}\pi$.

Problem 8

Use Green's Theorem to evaluate the integral $\oint_C x^2y\,dx - 3y^2\,dy$

where C is the circle $x^2+y^2=1$, oriented counter-clockwise.
ANSWER:

A convenient form of Green's theorem is $\iint\limits_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = \oint_C (M\,dx + N\,dy)\,.$

Letting $M=x^2y$ and $N=3y^2$, the right side exactly matches what we are asked to evaluate. To fill in the left side we need $\frac{\partial N}{\partial x}=0$ and $\frac{\partial M}{\partial y}=x^2$. Then the left side becomes $\iint\limits_S (0-x^2)dA$ where S is the disk $x^2+y^2 \le 1$. In rectangular coordinates we have $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(-x^2)\,dy\,dx$. Evaluating the integral will be easier in polar coordinates:

$$\int_0^{2\pi}\int_0^1 (-(r\cos\theta)^2)\,r\,dr\,d\theta = -\int_0^{2\pi}\frac{1}{4}(\cos^2\theta)\,d\theta = -\frac{1}{8}\int_0^{2\pi}(1+\cos 2\theta)\,d\theta = -\frac{\pi}{4}.$$

Problem 9

For motion along the curve given parametrically by $\vec{r}(t) = t^2\vec{\imath} + \frac{2}{3}t^3\vec{\jmath} + t\vec{k}$:

(a) Find the velocity $\vec{v}(t)$ at the point where $t = 1$

ANSWER: $\vec{v}(t) = \vec{r}\,'(t) = 2t\vec{\imath} + 2t^2\vec{\jmath} + \vec{k}$, so $\vec{v}(1) = 2\vec{\imath} + 2\vec{\jmath} + \vec{k}$.

(b) Find the acceleration $\vec{a}(t)$ at the point where $t = 1$

ANSWER: $\vec{a}(t) = \vec{r}\,''(t) = \vec{v}\,'(t) = 2\vec{\imath} + 4t\vec{\jmath} + 0\vec{k}$, so $\vec{a}(1) = 2\vec{\imath} + 4\vec{\jmath}$.

(c) Find the unit tangent vector $\vec{T}(t)$ at the point where $t = 1$

ANSWER: $\vec{T}$ will be a unit vector in the direction of $\vec{v}$. Some methods for solving parts (d) and (e) would require we know $\vec{T}(t)$ as a function so we could take its derivative. I will do (d) and (e) another way, so we can just use $\vec{v}(1)$ and change its magnitude to 1. From (a), $|\vec{v}(1)| = \sqrt{4+4+1} = 3$. Hence $\vec{T}(1) = \frac{1}{3}(2\vec{\imath} + 2\vec{\jmath} + \vec{k}$ or $\frac{2}{3}\vec{\imath} + \frac{2}{3}\vec{\jmath} + \frac{1}{3}\vec{k}$.

(d) Find the principal unit normal vector $\vec{N}(t)$ at the point where $t = 1$

ANSWER: I find it easiest to calculate the tangential and normal components of acceleration and use those to get $\vec{N}$.

$$a_T = \frac{\vec{r}\,' \cdot \vec{r}\,''}{|\vec{r}\,'|} \quad \text{and} \quad a_N = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|}.$$

We already have $\vec{r}\,'(1) = \vec{v}(1) = 2\vec{\imath} + 2\vec{\jmath} + \vec{k}$ and $\vec{r}\,''(1) = \vec{a}(1) = 2\vec{\imath} + 4\vec{\jmath}$. Using these gives, for $t = 1$, $a_T = \frac{(2\vec{\imath}+2\vec{\jmath}+\vec{k})\cdot(2\vec{\imath}+4\vec{\jmath})}{3} = \frac{4+8}{3} = 4$, and $a_N = \frac{(2\vec{\imath}+2\vec{\jmath}+\vec{k})\times(2\vec{\imath}+4\vec{\jmath})}{3} = \frac{|-4\vec{\imath}+2\vec{\jmath}+4\vec{k}|}{3} = \frac{6}{3} = 2$.
Now from $\vec{a} = a_T\vec{T} + a_N\vec{N}$ we have $\vec{N}(1) = \frac{1}{a_N}\left(\vec{a} - a_t\vec{T}\right) = \frac{1}{2}\left((2\vec{\imath} + 4\vec{\jmath}) - 4(\frac{2}{3}\vec{\imath} + \frac{2}{3}\vec{\jmath} + \frac{1}{3}\vec{k})\right)$
$= -\frac{1}{3}\vec{\imath} + \frac{2}{3}\vec{\jmath} - \frac{2}{3}\vec{k}$.
(Note that you can check that somewhat by computing $|\vec{N}| = 1$ and $\vec{N}\cdot\vec{T} = 0$ so they are perpendicular!)

(e) Find the curvature $\kappa(t)$ at the point where $t = 1$

ANSWER:

Using the formula $\kappa = \dfrac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$ and the values calculated above, we get

$$\kappa = \frac{6}{3^3} = \frac{2}{9}.$$

Problem 10

Let $f(x,y) = \dfrac{xy}{x^2+y^2}$.

Show that

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist.
ANSWER: If we come toward the origin along the x-axis, where $y = 0$, we have $f(x,0) = \frac{0}{x^2}$. For any $x \neq 0$ we get $f(x,0) = 0$, so the limit coming in along the x-axis will be that constant, 0. We get the same limit coming along the y-axis, so we have to look further for evidence that the limit does not exist. You could use a formula like $y = mx$ to account for all (non-vertical) lines through the origin at once, but $m = 1$, the line $y = x$, turns out to be enough. Along that line we have $f(x,y) = f(x,x) = \frac{x^2}{2x^2}$ so for any $x \neq 0$ we have $f(x,x) = \frac{1}{2}$. Thus the limit coming to the origin along the line $y = x$ must be that constant, $\frac{1}{2}$.
Now we have two different approaches to $(0,0)$ yielding two different numbers as their candidates for the limit, but for the limit to exist all such approaches would have to agree. Hence the limit does not exist.