Mathematics 234, Fall 2005 Lecture 2 (Wilson)

First Midterm Exam October 6, 2004 ANSWERS

Problem 1

Let $f(x, y) = x\cos(y) + e^{xy}$ and let P be the point $(2, 0)$.

(a) Find the gradient $\overrightarrow{\nabla f}$ at P.

ANSWER:

We compute first the partial derivatives. $f_x = \cos(y) + y\,e^{xy}$ which when $x = 2$ and $y = 0$ gives $1 + 0 = 1$. $f_y = -x\sin(y) + x\,e^{xy}$ which at $(2, 0)$ gives $0 + 2 = 2$ Thus $\overrightarrow{\nabla f}$ at $(2, 0)$ is $\vec{\imath} + 2\vec{\jmath}$.

(b) For the vector $\vec{v} = -3\vec{\imath} + 4\vec{\jmath}$, find the directional derivative of f at P in the direction of $\vec{v}$.

ANSWER:

First we find a unit vector $\vec{u}$ in the direction of $\vec{v}$. Since $|\vec{v}| = \sqrt{9 + 16} = 5$, we use $\vec{u} = \frac{1}{5}\vec{v} = -\frac{3}{5}\vec{\imath} + \frac{4}{5}\vec{\jmath}$. Now compute the dot product of that with the gradient from (a) and get $-\frac{3}{5} + \frac{8}{5} = 1$.

(c) Find a vector $\vec{w}$ in the direction of the level curve $f(x, y) = 3$ at P.

ANSWER:

We know the gradient is perpendicular to the level curve. Hence we can use any vector perpendicular to the gradient. If we let $\vec{w}$ have the components from the gradient, but with the order interchanged and one sign changed, i.e. $\vec{w} = -2\vec{\imath} + \vec{\jmath}$, then its dot product with the gradient will be zero so they will be perpendicular.

(d) Give a vector in the direction which makes the directional derivative at P largest. What is the directional derivative in that direction?

ANSWER:

The gradient points in the direction of the largest directional derivative, so we can use that vector: $\vec{\imath} + 2\vec{\jmath}$. In that direction the derivative is the magnitude of the gradient, $\sqrt{1 + 4} = \sqrt{5}$.

Problem 2

(a) The function $f(x, y) = \frac{x^2 - 2xy + 4y}{x^2 - y^2}$ is continuous at the point $(1, 2)$.
How can I tell that is true? Give reasons based on theorems in the textbook: You do not need to quote the theorems or cite where a theorem appears in the book.

ANSWER:

Since f is given as a quotient of polynomials, which are continuous everywhere, f will be continuous anywhere its denominator is not zero. This denominator gives $1^2 - 2^2 = 1 - 4 = -3$ at the given point, so it is not zero.

What is $\lim\limits_{(x,y)\to(1,2)} f(x, y)$?

ANSWER:

Since f is continuous at this point, the limit will be the value of f at the point, i.e. $\frac{1^2 - 2\times1\times2 + 4\times2}{1^2 - 2^2} = \frac{5}{-3}$ or $-\frac{5}{3}$.

(b) The function $f(x,y) = \frac{2xy}{x^2+y^2}$ does <u>not</u> have a limit as $(x,y) \to (0,0)$.
Justify this statement carefully: Give numeric evidence, don't just paraphrase the definition of the limit.

ANSWER:

If we approach $(0,0)$ along the line $y = mx$ for some constant m, we have $f(x,y) = f(x,mx) = \frac{2mx^2}{x^2+m^2x^2}$ which is equal to $\frac{2m}{1+m^2}$ for any $x \neq 0$. Hence the limit coming in along such a line is $\frac{2m}{1+m^2}$. Along different lines, i.e. for different values of m, this gives different results. For example, coming in along $y = 0$, where $m = 0$, we get 0, but along $y = x$ ($m = 1$) we get $\frac{2}{1+1} = 1$. Since we don't have a consistent limit on different lines there is no limit.

<u>Problem 3</u>
Some mechanism forces an object to move with position vector given by $\vec{r}(t) = t^2\vec{\imath} + e^t\vec{\jmath} + t\,e^t\vec{k}$, for $-1 \leq t \leq 1$.

(a) Find the velocity for this motion, both in general and at $t = 0$.

ANSWER:

$\vec{v}(t) = \vec{r}'(t) = 2t\vec{\imath} + e^t\vec{\jmath} + e^t(1+t)\vec{k}$. At $t = 0$ we get $\vec{v}(0) = \vec{\jmath} + \vec{k}$.

(b) Find the acceleration for this motion, both in general and at $t = 0$.

ANSWER:

$\vec{a}(t) = \vec{v}'(t) = 2\vec{\imath} + e^t\vec{\jmath} + e^t(2+t)\vec{k}$. At $t = 0$ we have $\vec{a}(0) = 2\vec{\imath} + \vec{\jmath} + 2\vec{k}$.

(c) Find equations for the tangent line to this object's path at $t = 0$.

ANSWER: The tangent line goes through the position given by $\vec{r}(0)$, i.e. $(0,1,0)$, in the direction of $\vec{v}(0) = \vec{\jmath} + \vec{k}$. In parametric form we can write $x = 0 + 0t$, $y = 1 + 1t$, and $z = 0 + 1t$, i.e. $x = 0$, $y = 1 + t$, and $z = t$.

(d) Suppose the mechanism breaks and from the instant $t = 0$ the object flies freely. Will it ever hit the plane where $y = z$? If so, at what point will it hit that plane?

ANSWER:

For the object, flying along the tangent line derived in (c), to hit $y = z$, we must have $1 + t = t$. Since that implies $1 = 0$, it never occurs.

<u>Problem 4</u>
Let $\vec{r}(t) = 12\sin(t)\vec{\imath} + 5t\,\vec{\jmath} + 12\cos(t)\vec{k}$ describe the motion of an object along a curve in space. Find as functions of t:

(a) The velocity $\vec{v}(t)$

ANSWER:

$\vec{v}(t) = \vec{r}'(t) = 12\cos(t)\vec{\imath} + 5\vec{\jmath} - 12\sin(t)\vec{k}$.

(b) The acceleration $\vec{a}(t)$

ANSWER:

$\vec{a}(t) = \vec{v}'(t) = -12\sin(t)\vec{\imath} - 12\cos(t)\vec{k}$.

(c) The unit tangent vector $\vec{T}(t)$

ANSWER:

The magnitude of $\vec{v}(t)$ is $\sqrt{144\cos^2(t)+25+144\sin^2(t)} = \sqrt{144+25} = \sqrt{169} = 13$, so $\vec{T}(t) = \frac{1}{13}\vec{v}(t) = \frac{12}{13}\cos(t)\vec{\imath} + \frac{5}{13}\vec{\jmath} - \frac{12}{13}\sin(t)\vec{k}$.

(d) The principal unit normal vector $\vec{N}(t)$

(e) The curvature $\kappa(t)$

I will compute a_T and a_N first, then do (d) and (e):

(f) The tangential (scalar) component of acceleration a_T

ANSWER:

$a_T = \frac{r' \cdot r''}{|r'|} = \frac{\vec{v}\cdot\vec{a}}{|\vec{v}|}$. Using our answers to (a) and (b) and the magnitude computed in (c), we have $a_T = \frac{-144\cos(t)\sin(t)+144\cos(t)\sin(t)}{13} = 0$.

(g) The normal (scalar) component of acceleration a_N

ANSWER:

Since $a_T = 0$, $\vec{a} = 0\vec{T} + a_N\vec{N}$, i.e. $\vec{N}$ is a unit vector in the direction of $\vec{a}$ and a_N is the magnitude of $\vec{a}$. Thus $a_N = |\vec{a}(t)| = \sqrt{144(\sin^2(t)+\cos^2(t)} = \sqrt{144} = 12$. Then $\vec{N} = \frac{1}{12}\vec{a}(t) = -\sin(t)\vec{\imath} - \cos(t)\vec{k}$.

One of the many formulas for the curvature $\kappa(t)$ is $\frac{a_N}{|r'|^2}$. Hence $\kappa(t) = \frac{12}{13^2} = \frac{12}{169}$ for all t.

Problem 5

For the function $f(x,y) = x^2 - xy + 3y^2$:

ANSWER:

Since it will help in both (a) and (b) I first note the linearization $L(x,y)$, using the partial derivatives when $x = 3$ and $y = 1$, is $L(x,y) = 15 + 7(x-3) - 9(y+1)$.

(a) Find an equation for the tangent plane to the graph of $f(x,y)$ at the point $(3,-1,15)$.

ANSWER:

Since the tangent plane is the graph of $L(x,y)$, we can use $z = 15 + 7(x-3) - 9(y+1)$.

(b) Use a linear approximation to estimate $f(2.98,-0.97)$.

ANSWER:

We compute $L(2.98,-0.97) = 15 + 7(-0.02) - 9(0.03) = 15 - .41 = 14.59$.
(If we compute the actual value $f(2.98,-0.97)$ we get 14.5937, so this is pretty good!)

Problem 6

Let $f(x,y,z) = xyz + x\sin(z)$.

(a) Calculate all three first partial derivatives of f.

ANSWER:

$\frac{\partial f}{\partial x} = yz + \sin(z)$, $\frac{\partial f}{\partial y} = xz$, and $\frac{\partial f}{\partial z} = xy + x\cos(z)$.

(b) Calculate all nine second partial derivatives of f.

ANSWER:
$\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial z^2} = -x\sin(z)$, $\frac{\partial^2 f}{\partial y \partial x} = z$, $\frac{\partial^2 f}{\partial z \partial x} = y + \cos(z)$, $\frac{\partial^2 f}{\partial x \partial y} = z$, $\frac{\partial^2 f}{\partial z \partial y} = x$, $\frac{\partial^2 f}{\partial x \partial z} = y + \cos(z)$, and $\frac{\partial^2 f}{\partial y \partial z} = x$.
You could save some work by using and citing the fact that the function is so "nice" that it has continuous second partial derivatives, and hence the mixed partials can be computed in either order. By calculating them all individually, as above, we can instead use that fact as a check on our answers.

Problem 7
Let $f(x,y) = x^2 + 2x + y^2 - 3y + 4$.
Find all points (x,y) such that the tangent plane to the surface $z = f(x,y)$ at the point $(x,y,f(x,y))$ is parallel to the plane $-2x + 3y - z = 0$.
Hint: Write an equation involving x_0 and y_0 that describes the tangent plane to the surface at the particular point $(x_0, y_o, f(x_0, y_0))$. Remember how the equations of parallel planes compare, and solve for which value(s) of x_0 and y_0 make these parallel.
ANSWER:
Using the hint, we first find the tangent plane. We have had two ways to do this, either works, but I will let $F(x,y,z) = f(x,y) - z = x^2 + 2x + y^2 - 3y + 4 - z$ and then the surface is $F(x,y,z) = 0$. The gradient of F at (x_0, y_0, z_0) is $(2x_0 + 2)\vec{\imath} + (2y_0 - 3)\vec{\jmath} - \vec{k}$, which must be perpendicular to the tangent plane at (x_0, y_0, z_0). But the vector $-2\vec{\imath} + 3\vec{\jmath} - \vec{k}$ is perpendicular to the given plane, so for the tangent plane to be parallel to the given plane these two vectors must be parallel. Hence for some number C, $2x_0 + 2 = -2C$, $2y_0 - 3 = 3C$, and $1 = 1 \times C$. From the last of these we get $C =$, so $2x_0 + 2 = -2$, $x_0 = -2$, and $2y_0 - 3 = 3$, $y_0 = 3$. Thus the only point is $(-2, 3)$. (Since $f(-2,3) = 4$, the actual point on the surface is $(-2,3,4)$. But we were asked for the x and y values.)

Problem 8

(a) Consider the function given in cylindrical coordinates $F(r,\theta,z) = 2r^2 + z^2$. Convert the function to rectangular (x-y-z) coordinates and find an equation for the tangent plane to the level surface through the point (in rectangular coordinates) (1,2,3).

ANSWER:

Since $r^2 = x^2 + y^2$, the function becomes $f(x,y,z) = 2x^2 + 2y^2 + z^2$ in rectangular coordinates.

Then $f(1,2,3) = 2 + 8 + 9 = 19$, so the level surface in question is $f(x,y,z) = 19$. The gradient of f at $(1,2,3)$ is $4\vec{\imath} + 8\vec{\jmath} + 6\vec{k}$, which must be perpendicular to the tangent plane. The plane goes through $(1,2,3)$ so its equation can be written $4(x-1) + 8(y-2) + 6(z-3) = 0$.

(b) An object moves in air so that its position (x,y,z) at time t is given by $(\sin(t), 2\cos(t), e^{3t})$.

The region of space through which the object moves has air temperature given by $T(x,y,x) = 20 + xyz - x^2 - 2y$ degrees on some scale.

Find $\frac{dT}{dt}$, both for t in general and at the instant when $t = \pi$.

ANSWER:

Using the chain rule, $\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$.

Computing the derivatives of the given functions we have $\frac{\partial T}{\partial x} = yz - 2x$, $\frac{\partial T}{\partial y} = xz - 2$, and $\frac{\partial T}{\partial z} = xy$, while $\frac{dx}{dt} = \cos(t)$, $\frac{dy}{dt} = -2\sin(t)$, and $\frac{dz}{dt} = 3e^{3t}$.

Putting these together we get $\frac{dT}{dt} = (yz - 2x)\cos(t) - 2(xz - 2)\sin(t) + 3(xy)e^{3t}$. You can reduce all of those to explicit functions of t or leave them in that form.

At $t = \pi$ we have $x = 0$, $y = -2$, and $z = e^{3\pi}$. So the derivative at that time is $2\,e^{3\pi}$.