Final Exam December 22, 2004 ANSWERS

Problem 1

For the helical ("corkscrew") motion $\vec{r}(t) = 2\cos(t)\vec{\imath} - 2\sin(t)\vec{\jmath} + 3t\vec{k}$:

(a) Find the velocity $\vec{v}(t)$ as a function of t.

ANSWER: $\vec{v}(t) = \vec{r}\,'(t) = -2\sin(t)\vec{\imath} - 2\cos(t)\vec{\jmath} + 3\vec{k}$.

(b) Find the acceleration $\vec{a}(t)$ as a function of t.

ANSWER: $\vec{a}(t) = \vec{v}\,'(t) = -2\cos(t)\vec{\imath} + 2\sin(t)\vec{\jmath} + 0\vec{k}$.

(c) Find the unit tangent vector $\vec{T}(t)$.

ANSWER: $\vec{T}(t)$ is the unit vector in the direction of $\vec{v}(t)$. The magnitude of $\vec{v}(t)$, using part (a), is $|\vec{v}(t)| = \sqrt{4\sin^2(t) + 4\cos^2(t) + 9} = \sqrt{13}$. Hence $\vec{T}(t) = \frac{1}{\sqrt{13}}\vec{v}(t) = -\frac{2}{\sqrt{13}}\sin(t)\vec{\imath} - \frac{2}{\sqrt{13}}\cos(t)\vec{\jmath} + \frac{3}{\sqrt{13}}\vec{k}$.

(d) Find the principal unit normal vector $\vec{N}(t)$.

ANSWER: $\vec{N}(t) = \frac{d\vec{T}/ds}{|d\vec{T}/ds|}$ and $\frac{d\vec{T}}{ds} = \frac{\vec{T}\,'(t)}{|\vec{v}(t)|}$. We got $|\vec{v}(t)| = \sqrt{13}$ in part (c). Also from part (c) we have $\vec{T}(t)$, so we can calculate $\vec{T}\,'(t) = -\frac{2}{\sqrt{13}}\cos(t)\vec{\imath} + \frac{2}{\sqrt{13}}\sin(t)\vec{\jmath} + 0\vec{k}$. Then $\frac{d\vec{T}}{ds} = \frac{1}{\sqrt{13}}\left(-\frac{2}{\sqrt{13}}\cos(t)\vec{\imath} + \frac{2}{\sqrt{13}}\vec{\jmath}\right) = -\frac{2}{13}\cos(t)\vec{\imath} + \frac{2}{13}\sin(t)\vec{\jmath}$. Now we can calculate $|d\vec{T}/ds| = \frac{2}{13}$, and $\vec{N}(t) = \frac{13}{2}\left(-\frac{2}{13}\cos(t)\vec{\imath} + \frac{2}{13}\sin(t)\vec{\jmath}\right) = -\cos(t)\vec{\imath} + \sin(t)\vec{\jmath} + 0\vec{k}$.

You might also see this answer geometrically with almost no calculation, in several ways: (i) Since the motion in the $\vec{k}$ direction has constant speed, the normal vector will point directly inward from $\vec{r}(t)$ toward the z-axis. Hence its $\vec{k}$ component will be zero and its $\vec{\imath}$ and $\vec{\jmath}$ components will be just the negatives of what appear in $\vec{r}$ except for normalizing to a unit vector. (ii) Since the "speed" $|\vec{v}|$ is a constant, $\sqrt{13}$, the acceleration must be entirely in the direction of the normal vector $\vec{n}(t)$. Hence we could take $\vec{a}(t)$ from (b) and multiply by a constant to get unit length. (iii) You can also realize that (i) or (ii) applies if you notice that $\vec{a}$ and $\vec{r}$ are the same except for the $\vec{k}$ component.

BUT: A common incorrect procedure involved dividing by a vector. No matter what else you are doing, that should raise a waving red flag to say something is wrong!

(e) Find the curvature $\kappa(t)$.

ANSWER: $\kappa(t) = \left|\frac{d\vec{T}}{ds}\right|$ so using what we got in (d) we have $\kappa(t) = \frac{2}{13}$.

One error that you should not have made was to get a vector as an answer: The curvature is a number, and furthermore it is never negative.

Problem 2

The function $f(x,y) = \dfrac{2xy^2}{x^2 + y^4}$ does not have a limit as $(x,y) \to (0,0)$.

Show that this is true.

ANSWER: Since we are told the limit does not exist, we look for paths such that approaching $(0,0)$ along different paths produces different apparent limits. There are many possible paths to choose. Here is one way to do it "all at once": Suppose we approach $(0,0)$ along the curve $x = ay^2$ for some number a, a parabola if $a \neq 0$ and the y-axis if $a = 0$. Where $x = ay^2$, but $(x,y) \neq (0,0)$, $f(x,y) = \frac{2a\,y^4}{a^2y^4+y^4} = \frac{2a}{a^4+1}$. That is a constant as $(x,y) \to (0,0)$ and hence the limit along the path is that constant. But choosing different values for a gives different values for the limit, e.g. $a = 1$ gives $\frac{2}{2} = 1$ while $a = 2$ gives $\frac{4}{17} \neq 1$.

Problem 3

Let $f(x,y) = x^2y + e^{xy}\sin y$.

(a) What is the gradient ∇f as a function of x and y?

ANSWER: $\dfrac{\partial f}{\partial x} = 2xy + ye^{xy}\sin y$ and $\dfrac{\partial f}{\partial y} = x^2 + xe^{xy}\sin y + e^{xy}\cos y$.

Thus $\nabla f(x,y) = (2xy + ye^{xy}\sin y)\,\vec{\imath} + (x^2 + xe^{xy}\sin y + e^{xy}\cos y)\,\vec{\jmath}$

(b) At the point $(1,0)$, in what direction $\vec{u}$ is the directional derivative $D_{\vec{u}}f$ largest? In what direction is the directional derivative smallest?

ANSWER: At $(1,0)$ the gradient, from (a), is $0\vec{\imath} + 2\vec{\jmath} = 2\vec{\jmath}$. That is not a unit vector, so we use the unit vector $\vec{u} = \vec{\jmath}$ in the same direction to specify the direction of the greatest directional derivative at $(1,0)$. If we use $\vec{u}$ in the opposite direction, $\vec{u} = -\vec{\jmath}$, we will get the least directional derivative at the same point.

(c) What is the value of the directional derivative at $(1,0)$, in the direction making the directional derivative largest?

ANSWER: The value of the directional derivative in the direction of the gradient at a point is the magnitude of the gradient at that point, $|\nabla f(1,0)| = \sqrt{0^2 + 2^2} = 2$.

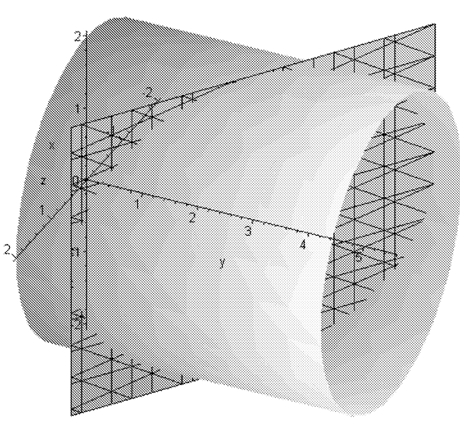
Problem 4

Set up but do not evaluate an iterated integral for the integral of

$$f(x,y,z) = 3x^z - 2z\cos(xy)$$

over the region in space which is inside the cylinder $x^2 + z^2 = 4$ and between the planes $y = 0$ and $x + y = 3$.

ANSWER:



The cylinder, since $x^2 + z^2 = 4$ makes no mention of y, is parallel to the y-axis. It intersects the xz-plane in a circle of radius 2, centered at the origin. The plane $x + y = 3$ extends vertically through the line $x + y = 3$ in the xy-plane, and cuts through the cylinder at 45°. See the picture to the right.

You could set this integral up in different orders for the variables. Using $dz\,dy\,dx$: The largest value of x, overall, is 2, and the smallest is -2. Hence the outermost integral will go from -2 to 2. For any x between -2 and 2, y goes from 0 to $3-x$ since the region is bounded by the planes $y=0$ and $y=3-x$ (rewritten version of $x+y=3$.) (When $x=-2$, y ranges from 0 to $3-(-2)=5$, along the back edge. On the front edge, $x=2$ and y ranges from 0 to $3-2=1$.) At any pair of values for x and y, z ranges from the lower half of the cylinder to the upper half, i.e. $-\sqrt{4-x^2}\le z\le\sqrt{4-x^2}$. Hence we can write the integral as

$$\int_{-2}^{2}\int_{0}^{3-x}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(3x^z-2z\cos(xy)\right)\,dz\,dy\,dx.$$

<u>Problem 5</u>

$$\text{For }\int_0^2\int_0^{\sqrt{4-x^2}} xy\sqrt{x^2+y^2}\,dy\,dx:$$

Evaluate the integral by converting to polar coordinates and evaluating the resulting polar integral.
ANSWER: The region of integration is a quarter-circle, the portion of the circle of radius 2 centered at the origin that is in the first quadrant where $x\ge 0$ and $y\ge 0$. In polar coordinates it can be described by $0\le r\le 2$ and $0\le\theta\le\frac{\pi}{2}$. That takes care of the limits on the integrals. The integrand, $xy\sqrt{x^2+y^2}$, can be converted using $x=r\cos\theta$, $y=r\sin\theta$, and $r=\sqrt{x^2+y^2}$. We must remember to replace $dy\,dx$ by $r\,dr\,d\theta$. Doing all of these things, we get

$$\int_0^{\frac{\pi}{2}}\int_0^2 (r\cos\theta)(r\sin\theta)(r)(r)\,dr\,d\theta \quad\text{or}\quad \int_0^{\frac{\pi}{2}}\int_0^2 r^4(\cos\theta)(\sin\theta)\,dr\,d\theta.$$

We can "pull" the $(\cos\theta)(\sin\theta)$ outside of the dr integral, and also replace it by $\frac{1}{2}\sin(2\theta)$ using an identity you were given on the exam, to get

$$\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin(2\theta)\int_0^2 r^4\,dr\,d\theta=\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin(2\theta)\left[\frac{r^5}{5}\right]_0^2 d\theta=\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin(2\theta)\frac{32}{5}\,d\theta=\frac{16}{5}\int_0^{\frac{\pi}{2}}\sin 2\theta\,d\theta$$

$$=\frac{8}{5}\left[-\cos 2\theta\right]_0^{\frac{\pi}{2}}=\frac{16}{5}.$$

<u>Problem 6</u>
If x, y, and z satisfy $z^3-xy+yz+y^3=2$:

(a) Find $\dfrac{\partial z}{\partial x}$.

ANSWER: It is not practical to solve the given equation for z. We find the derivatives implicitly. Taking $\frac{\partial}{\partial x}$ across the equation and remembering the chain rule, we get $3z^2\frac{\partial z}{\partial x}-y+y\frac{\partial z}{\partial x}=0$. Grouping on the left the terms that do include $\frac{\partial z}{\partial x}$ and on the right those that don't and factoring gives $\frac{\partial z}{\partial x}(3z^2+y)=y$, and dividing we get $\frac{\partial z}{\partial x}=\frac{y}{3z^2+y}$.

(b) Find $\dfrac{\partial z}{\partial y}$.

ANSWER: Proceeding in the same way except that we differentiate with respect to y, we have (remember the product rule this time as well as the chain rule!) $3z^2\frac{\partial z}{\partial y}-x+y\frac{\partial z}{\partial y}+z+3y^2=0$, so $\frac{\partial z}{\partial y}(3z^2+y)=x-z-3y^2$ and $\frac{\partial z}{\partial y}=\frac{x-z-3y^2}{3z^2+y}$.

(c) Find $\dfrac{\partial z}{\partial x}$ at the point $(1,1,1)$.

ANSWER: We just use $y = 1$ and $z = 1$ ($x = 1$ happens not to matter) in the expression $\frac{y}{3z^2+y}$ we got in (a), giving $\frac{1}{4}$.

Problem 7

Let $f(x, y, z) = xy + y + z$. Let C be the curve $\vec{r}(t) = 2t\vec{\imath} + t\vec{\jmath} + (2 - 2t)\vec{k}$ for $0 \le t \le 1$.

Evaluate the line integral $\displaystyle\int_C f(x, y, z)\, ds$.

ANSWER: This is a straightforward line integral. Along the curve, $x = 2t$ and $y = t$ and $z = 2 - 2t$, so $f(x, y, z) = (2t)(t) + t + (2 - 2t) = 2t^2 - t + 2$. $x' = 2$, $y' = 1$, and $z' = -2$, so $\sqrt{(x')^2 + (y')^2 + (z')^2} = \sqrt{4 + 1 + 4} = 3$. Hence we can evaluate the line integral as the ordinary integral

$$\int_0^1 (2t^2 - t + 2)(3)\, dt = 3\left[\frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t\right]_0^1 = 3\left(\frac{2}{3} - \frac{1}{2} + 2\right) = \frac{13}{2}.$$

Problem 8

One of the following two vector fields is conservative and the other is not.

$$\vec{F_1}(x, y, z) = (2x - 3)\vec{\imath} - z\vec{\jmath} + (\cos z)\vec{k}$$

$$\vec{F_2}(x, y, z) = (e^x \cos y + yz)\vec{\imath} + (xz - e^x \sin y)\vec{\jmath} + (xy + z)\vec{k}$$

(a) Which vector field is conservative? Which one is not conservative?

ANSWER: To test whether a field $\vec{F} = M\vec{\imath} + N\vec{\jmath} + P\vec{k}$ is conservative we check whether $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$, and $\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$.

For the field $\vec{F_1}$ where $M = 2x - 3$, $N = -z$, and $P = \cos z$, these amount to $0 = 0$, $0 = 0$, and $-1 = 0$. These are not all true so $\vec{F_1}$ is not conservative. From this we can infer $\vec{F_2}$ is conservative, but the problem as given on the exam said to test each explicitly. So we repeat the check for $\vec{F_2}$.

For $\vec{F_2}$: $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ both give $z - e^x \sin y$, $\frac{\partial M}{\partial z}$ and $\frac{\partial P}{\partial x}$ both give y, and $\frac{\partial N}{\partial z}$ and $\frac{\partial P}{\partial y}$ both give x. Hence $\vec{F_2}$ is a conservative field.

(b) For the vector field F that you found to be conservative, evaluate

$$\int_C \vec{F}(x, y, z) \cdot d\vec{r}$$

where C is any path leading from $(0, 0, 0)$ to $(-1, \frac{\pi}{2}, 2)$.

Note: As printed on the exam, the integral read

$$\int_C F(x, y, z)\, ds.$$

This was corrected during the exam to the verstion given above. This should not have caused confusion, the printed version really could not mean anything but $\vec{F}(x, y, z) \cdot d\vec{r}$ in context. The change was intended to make the problem match the formulas we had seen. In particular

Theorem A on page 742 in the text exactly matches the problem in this form. If you did the problem without the change you must have interpreted it to mean exactly this anyway!

ANSWER: The field to use is $\vec{F}_2$. We do not need to know what the curve is, if we can find a potential function $f(x, y, z)$ for the field. Once we have f we can simply evaluate $f(-1, \frac{\pi}{2}, 2) - f(0, 0, 0)$.

To find f: We know $\nabla f = \vec{F}_2$, so $\frac{\partial f}{\partial x} = M = e^x \cos y + yz$. Thus f must be of the form $e^x \cos y + xyz + g(y, z)$ where $g(y, z)$ denotes some part that can vary with y and z but not x. We also have $\frac{\partial f}{\partial y} = N = xz - e^x \sin y$: From what we had so far, $\frac{\partial f}{\partial y} = -e^x \sin y + xz + \frac{\partial g}{\partial y}$, so if we let $\frac{\partial g}{\partial y} = 0$ we are still OK. That means $g(y, z)$ might depend on z or might be constant, but it does not vary with y. As our last step we need $\frac{\partial f}{\partial z} = P = xy + z$. From what we have so far, where $f = e^x \cos y + xyz +$(some function $g(z)$ depending at most on z), $\frac{\partial f}{\partial z} = xy + \frac{\partial g}{\partial z}$, so $\frac{\partial g}{\partial z}$ must be z: We can achieve that if $g(z) = \frac{1}{2}z^2$. Putting it all together, $f(x, y, z) = e^x \cos y + xyz + \frac{1}{2}z^2$ is a potential function for $\vec{F}_2$. (We could add a constant and get another potential function, but we only need to find a potential function so this is sufficient. If we did add a constant it would occur with both + and − signs in the next calculation and hence it would have no effect on the answer.)

Now we evaluate $f(-1, \frac{\pi}{2}, 2) = 2 - \pi$ and $f(0, 0, 0) = 1$, and the integral is $2 - \pi - 1 = 1 - \pi$.

Problem 9

Evaluate $\oint_C -y^2\, dx + xy\, dy$

around the square in the xy-plane with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$, in that order.

ANSWER: You could parametrize each of the four legs of the curve C and do this as a line integral. It is much easier to use Green's Theorem and convert this to a double integral over the square that C is the boundary of. Since C is traversed in a counter clockwise direction we don't even need to worry about changing a sign. Viewing

$$\oint_C -y^2\, dx + xy\, dy \quad \text{as} \quad \oint_C M\, dx + N\, dy$$

where $M = -y^2$ and $N = xy$, Green's theorem says

$$\oint_C M\, dx + N\, dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

In this case $\frac{\partial N}{\partial x} = y$ and $\frac{\partial M}{\partial y} = -2y$, so we have to evaluate $\iint_S (y - (-2y))dA$ where S is the square $0 \le x \le 1$ and $0 \le y \le 1$,

$$\int_0^1 \int_0^1 (3y) dx\, dy = \int_0^1 (3y) dy = \left[\frac{3y^2}{2} \right]_0^1 = \frac{3}{2}.$$

If you do it as a line integral, you really need to have four pieces, not three. The problem says "around the square" and squares have four sides! So your path should go back to the starting point. But this is not an integral of a conservative field, so going around a closed path does not imply the result is zero, and as we saw above it is in fact not zero.

Problem 10

Evaluate $\oint_C \vec{F} \cdot \vec{T}\, ds$ where

$\vec{F}(x, y, z) = y\vec{\imath} + xz\vec{\jmath} + x^2\vec{k}$ and C is the boundary of the triangle in the plane $x + y + z = 1$ with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, traversed counterclockwise as viewed from above.

ANSWER: I will use Stokes' Theorem: It is also possible to do the problem directly, parametrizing the curve C in three parts and adding the results. Letting S be the triangle whose boundary is C, the theorem tells us

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_S (curl\vec{F}) \cdot \vec{n}\, dS$$

where $\vec{n}$ is a normal vector oriented consistently with the right-hand-rule for traversing C in the prescribed direction. Since C and S lie in the plane $x + y + z = 1$, a vector perpendicular to the plane is $\vec{\imath} + \vec{\jmath} + \vec{k}$. Since we go around C counterclockwise, the right-hand-rule indicates an upward-pointing vector, so this vector has the correct orientation. But it has length $\sqrt{3}$. So we let $\vec{n} = \frac{1}{\sqrt{3}}\vec{\imath} + \frac{1}{\sqrt{3}}\vec{\jmath} + \frac{1}{\sqrt{3}}\vec{k}$.
We compute $curl\vec{F}$ and get $-x\vec{\imath} - 2x\vec{\jmath} + (z - 1)\vec{k}$. Taking the dot product of that with $\vec{n}$ we get $\frac{1}{\sqrt{3}}(-3x + z - 1)$.
Now we need to compute the surface integral of $\frac{1}{\sqrt{3}}(-3x + z - 1)$ over the triangle.

We do this as $$\iint_R \frac{1}{\sqrt{3}}(-3x + z - 1)\sqrt{f_x^2 + f_y^2 + 1}\, dA$$

where f_x and f_y come from the expression of the surface in the form $z = f(x, y)$, which in this case is $z = 1 - x - y$. So $f_x = -1$ and $f_y = -1$, and R is the triangle in the xy-plane underneath the surface S. Thus we have the integral

$$\iint_R \frac{1}{\sqrt{3}}(-3x + (1 - x - y) - 1)\sqrt{3}\, dA = \iint_R (-4x - y)\, dA.$$

Finally, we set that up as an iterated integral. R is the triangle in the plane with vertices the origin, $(1, 0)$, and $(0, 1)$, which we can describe as $0 \leq x \leq 1$ and $0 \leq y \leq 1 - x$. The integral in iterated form is

$$\int_0^1 \int_0^{1-x} (-4x - y)\, dy\, dx = \int_0^1 \left[-4xy - \frac{y^2}{2}\right]_{y=0}^{y=1-x} dx = \int_0^1 \left(\frac{7}{2}x^2 - 3x - \frac{1}{2}\right) dx$$

$$= \left[\frac{7}{6}x^3 - \frac{3}{2}x^2 - \frac{1}{2}x\right]_0^1 = -\frac{5}{6}.$$