Mathematics 234, Fall 2004 Lecture 3 (Wilson)

First Midterm Exam October 4, 2004 ANSWERS

Problem 1

In class we saw pictures of the helix (corkscrew) given by the position vector

$$\vec{r}(t) = \sin(t)\vec{\imath} + t\,\vec{\jmath} + \cos(t)\vec{k}.$$

Calculate the arclength of one turn of this helix, from $t = 0$ to $t = 2\pi$.

ANSWER: The velocity vector $\vec{v}(t)$ is $\cos(t)\vec{\imath}+\vec{\jmath}-\sin(t)\vec{k}$, so $|\vec{v}|$ is $\sqrt{\cos^2(t) + 1 + \sin^2(t)} = \sqrt{2}$ at any value of t. Thus the arclength is

$$\int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\pi.$$

Problem 2

Let $\vec{r}(t) = 3\sin(t)\vec{\imath} + 3\cos(t)\vec{\jmath} + 4t\,\vec{k}$ describe the motion of an object along a curve in space. Find as functions of t:

(a) The velocity $\vec{v}(t)$

(b) The acceleration $\vec{a}(t)$

(c) The unit tangent vector $\vec{T}(t)$

(d) The principal unit normal vector $\vec{N}(t)$

(e) The curvature $\kappa(t)$

(f) The tangential (scalar) component of acceleration a_T

(g) The normal (scalar) component of acceleration a_N

ANSWERS: Differentiating,

$$\vec{v}(t) = 3\cos(t)\vec{\imath} - 3\sin(t)\vec{\jmath} + 4\vec{k}$$

and

$$\vec{a}(t) = -3\sin(t)\vec{\imath} - 3\cos(t)\vec{\jmath} + 0\vec{k}.$$

To find $\vec{T}$, we need a vector of unit length in the direction of $\vec{v}(t)$. The magnitude of $\vec{v}(t)$ is $|\vec{v}(t)| = \sqrt{9\cos^2(t) + 9\sin^2(t) + 16} = \sqrt{25} = 5$. Thus

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{1}{5}\vec{v}(t) = \frac{3}{5}\cos(t)\vec{\imath} - \frac{3}{5}\sin(t)\vec{\jmath} + \frac{4}{5}\vec{k}.$$

There are several ways to proceed to $\vec{N}$. One way is to go ahead and find the components of acceleration, a_T and a_N, and since we need them anyway I will do that. For the tangential component we have

$$a_T(t) = \vec{a}(t) \cdot \vec{T}(t) = -3\sin(t) \times \frac{3}{5}\cos(t) + (-3\cos(t)) \times (-\frac{3}{5}\sin(t)) + 0 \times \frac{4}{5} = 0.$$

Since $\vec{a}(t) = a_T\vec{T}(t) + a_N\vec{N}(t) = a_N\vec{N}(t)$, where $|\vec{N}(t)| = 1$, we have (i) $\vec{N}(t)$ is a unit vector in the direction of $\vec{a}(t)$ and (ii) a_N is the magnitude of $\vec{a}(t)$. Hence $a_N = \sqrt{9\sin^2(t) + 9\sin^2(t)} = \sqrt{9} = 3$, and $\vec{N}(t) = \frac{1}{3}\vec{a}(t) = -\sin(t)\vec{\imath} - \cos(t)\vec{\jmath} + 0\vec{k}$.

This leaves only $\kappa(t)$ to be determined. Again there are several ways to find it: Given what we have already computed, it may be easiest to compute

$$\kappa(t) = \frac{a_N}{|\vec{v}(t)|^2} = \frac{3}{25}$$

which happens not to depend on t.

Problem 3

For each of the four descriptions, fill in A or B or C or D to indicate which graph below it corresponds to.

(i) $z = y\sin(x)$

ANSWER: Notice that figure C has a sine-wave cross-section in one direction, like $z = \sin(x)$, while in the other horizontal direction it has cross sections that are straight lines, like $z =$(some constant)y. C is the answer here.

(ii) $r = 1 + \cos(\theta)$, any value of z

ANSWER: From the variables r, θ, and z, we can infer this is in cylindrical coordinates. Since z does not appear in the equation, if any point is on the surface so is the entire vertical line through that point. That alone is enough to pick out figure D. When you also notice that a horizontal cross-section will be $r = 1 + \cos\theta$ in polar coordinates, a cardioid, you have confirming evidence.

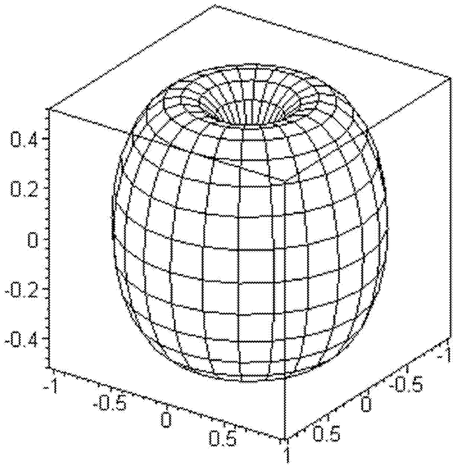
(iii) $z = 3 - x^2 - y$

ANSWER: This surface slopes downward as y increases, and upward in the $-y$ direction, and follows the downward opening parabola $z = 3 - x^2$ as you move in the $\pm x$ directions along the x-axis, and surface B certainly fits this description.

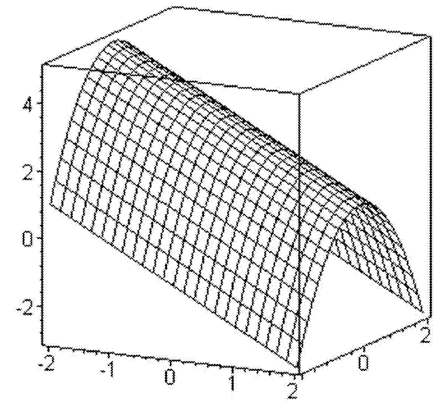
(iv) $\rho = \sin(\phi)$, any value of θ

ANSWER: This must be in spherical coordinates, and since θ does not appear in the equation it will be completely symmetric around the z axis. That picks out surface A. In addition, for any fixed θ, the cross-section will be $\rho = \sin\phi$: $r = \sin\theta$ gave a circle in polar coordinates, so looking out from the z-axis in any direction we should see a circle: This confirms the "donut" shape in figure A.

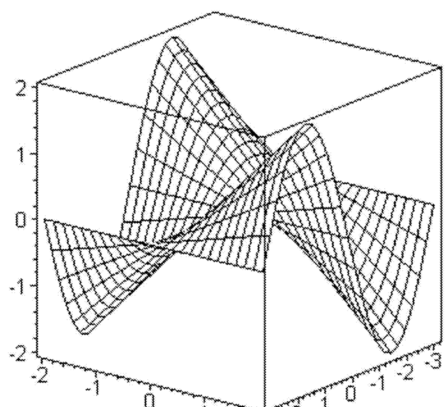
A



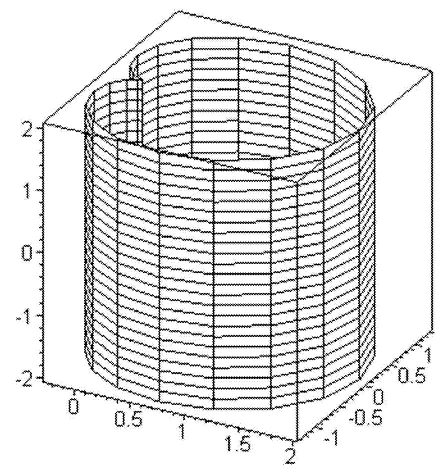
B



C



D



Problem 4

For $f(x,y) = \sin(x^2+2y)$, find:

(a) $\dfrac{\partial f}{\partial x}$

ANSWER: Remembering the chain rule, $\frac{\partial f}{\partial x} = \cos(x^2+2y) \times 2x = 2x\cos(x^2+2y)$.

(b) $\dfrac{\partial f}{\partial y}$

ANSWER: Similarly, $\frac{\partial f}{\partial y} = \cos(x^2+2y) \times 2 = 2\cos(x^2+2y)$.

(c) $\dfrac{\partial^2 f}{\partial x^2}$

ANSWER: We had $\frac{\partial f}{\partial x} = 2x\cos(x^2+2y)$: we differentiate that with respect to x, using the product and chain rules, we get $\frac{\partial^2 f}{\partial x^2} = 2\cos(x^2+2y) - 4x^2\sin(x^2+2y)$.

(d) $$\frac{\partial^2 f}{\partial y \partial x}$$

ANSWER: Since this says to take the derivative first with respect to x and then with respect to y, we take the derivative of $\frac{\partial f}{\partial x} = 2x\cos(x^2+2y)$ with respect to y. Using the chain rule we get $\frac{\partial^2 f}{\partial y \partial x} = -4x\sin(x^2+2y)$.

(e) $$\frac{\partial^2 f}{\partial x \partial y}$$

ANSWER: We can cite the theorem that says if the partials are all continuous then the mixed second partials will be equal, so the answer should be the same as for (d). Or we can directly compute the derivative of $2\cos(x^2+2y)$ with respect to x. Either way, we get $\frac{\partial^2 f}{\partial x \partial y} = -4x\sin(x^2+2y)$.

(f) $$\frac{\partial^2 f}{\partial y^2}$$

ANSWER: Differentiating $\frac{\partial f}{\partial y} = 2\cos(x^2+2y)$ with resepect to y we get $\frac{\partial^2 f}{\partial y^2} = -4\sin(x^2+2y)$.

Problem 5

(a) Evaluate $\lim_{(x,y)\to(2,\pi)} x^2\cos(2y)$.

ANSWER: This function is constructed by multiplying two functions (x^2 and $\cos(2y)$) which are each continuous at all input values, so it is continuous at all points (x, y). Hence we can evaluate the limit by "plugging in" and $\lim_{(x,y)\to(2,\pi)} x^2\cos(2y) = 2^2\cos(2\pi) = 4$.

(b) Show that $\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$ does not exist.

ANSWER: If we approach $(0, 0)$ along the y-axis, where $x = 0$, the function is constantly 0 for all values of $y \neq 0$. Hence the limit is 0 as we approach the origin along the y-axis.

If we instead approach along the x-axis, where $y = 0$, the function values are $\frac{x}{\sqrt{x^2}}$. For $x > 0$ this evaluates to 1, so the limit coming in along the postive x-axis is 1, which is not the same as the 0 we got along the y-axis, and so the limit must not exist. In fact the limit coming in along the negative x-axis is actually -1, since for $x < 0$ we have $\sqrt{x^2} = |x| = -x$, so we even get different limits coming to zero along the x-axis from opposite sides, and so we did not actually need the y-axis result.

Problem 6

Let $f(x, y) = x^2 e^y$.

Let P be the point $(1, 0)$.

(a) Find the gradient ∇f at the point P.

ANSWER: The gradient at any point (x, y) is $\nabla f(x, y) = 2x\, e^y \vec{\imath} + x^2\, e^y \vec{\jmath}$, so when $x = 1$ and $y = 0$ we have $\nabla f(1, 0) = 2\vec{\imath} + \vec{\jmath}$.

(b) Find the directional derivative of f at P in the direction of an arbitrary vector $\vec{v} = v_1\vec{\imath} + v_2\vec{\jmath}$.

ANSWER: First find a unit vector in the direction of $\vec{v}$: Let $\vec{u} = \frac{1}{|\vec{v}|}\vec{v} = \frac{1}{\sqrt{v_1^2+v_2^2}} v_1\vec{\imath} + \frac{1}{\sqrt{v_1^2+v_2^2}} v_2\vec{\jmath}$. Now the directional derivative $D_{\vec{u}} f(1, 0) = \nabla f(1, 0) \cdot \vec{u} = \frac{2v_1+v_2}{\sqrt{v_1^2+v_2^2}}$.

(c) Find the directional derivative of f at P in the direction from $(1, 0)$ to $(4, 4)$.

ANSWER: In this case the vector $\vec{v} = 3\vec{\imath} + 4\vec{\jmath}$, where $v_1 = 3$ and $v_2 = 4$, is in the correct direction. Then $\sqrt{v_1^2 + v_2^2} = \sqrt{9 + 16} = 5$ and the answer to (b) becomes $\frac{2\times3+4}{5} = 2$.

(d) In what direction is the directional derivative of f at P largest? What is the directional derivative in that direction?

ANSWER: The direction of the gradient, $2\vec{\imath} + \vec{\jmath}$, gives the direction in which the directional derivative is greatest. In the form of a unit vector this is $\frac{2}{\sqrt{5}}\vec{\imath} + \frac{1}{\sqrt{5}}\vec{\jmath}$. As we have seen, the directional derivative in that direction will be the magnitude of the gradient, $\sqrt{5}$.

Problem 7

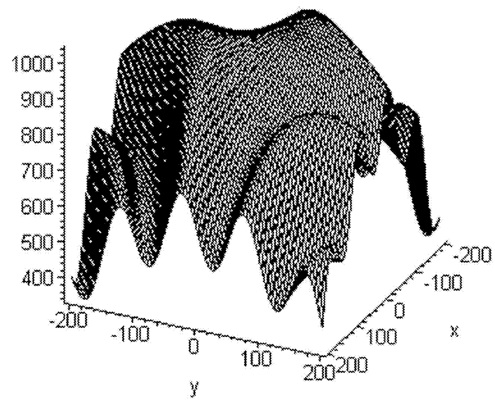
Suppose $w = \sin(xy) + x\sin(y)$, where $x = u^2 + v^2$ and $y = 2u + v - 2$. Using the chain rule:

(a) $$\frac{\partial w}{\partial u} = (y\cos(xy) + \sin(y)) \times 2u + (x\cos(xy) + x\cos(y)) \times 2.$$

(b) $$\frac{\partial w}{\partial v} = (y\cos(xy) + \sin(y)) \times 2v + (x\cos(xy) + x\cos(y)) \times 1.$$

Problem 8

Suppose you are walking over some hills. The north-south direction is measured as y and the east-west direction as x. The altitude in feet is given by



$$f(x,y) = 1000 - 100\sin\left(\frac{xy}{3000}\right) - \frac{x^2}{100}.$$

If you are at the point where $x = 100$ and $y = -100$, and you start walking toward the center where $x = 0$ and $y = 0$, will you begin by walking uphill, downhill, or horizontally? Be sure to show how you determine your answer.

<u>ANSWER</u>: We need to find the directional derivative for the given altitude function $f(x,y)$ in the direction of a vector from $(100,-100)$ toward $(0,0)$. A unit vector in that direction is $-\frac{1}{\sqrt{2}}\vec{\imath} + \frac{1}{\sqrt{2}}\vec{\jmath}$.
The partial derivatives of f are $\frac{\partial f}{\partial x} = -\frac{y}{30}\cos\left(\frac{xy}{3000}\right) - \frac{x}{50}$ and $\frac{\partial f}{\partial y} = -\frac{x}{30}\cos\left(\frac{xy}{3000}\right) - \frac{y}{100}$. At $(100,-100)$ we get $\frac{\partial f}{\partial x} = \frac{10}{3}\cos(-\frac{10}{3}) - 2$, and $\frac{\partial f}{\partial y} = -\frac{10}{3}\cos(-\frac{10}{3}) + 1$. Thus the directional derivative we seek is $-\frac{1}{\sqrt{2}}(\frac{10}{3}\cos(-\frac{10}{3}) - 2) + \frac{1}{\sqrt{2}}(-\frac{10}{3}\cos(-\frac{10}{3}) + 1)$ which simplifies to $-\frac{1}{\sqrt{2}}\left(\frac{20}{3}\cos(-\frac{10}{3}) - 3\right)$. With a calculator you can approximate that as $+6.748975979$, but all we need to know is whether it is positive or negative, and we can tell that without a calculator: $\frac{10}{3} = 3\frac{1}{3}$ is just a bit larger than π, so the point on the unit circle determined by $-\frac{10}{3}$ is just above the x-axis and nearly at $x = -1$. Thus $\cos(-\frac{10}{3})$ is negative (nearly -1) so both the terms in $\frac{20}{3}\cos(-\frac{10}{3}) - 3$ are negative,

so their combination must be < 0, and multiplying by $-\frac{1}{\sqrt{2}}$ will give us a positive number.
Hence the directional derivative is positive and our path leads uphill.