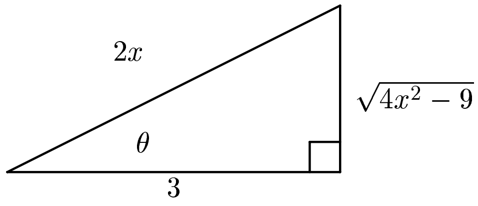
Problem 1 (15 points)
Evaluate the integrals:

(a) $$\int \frac{\sqrt{4x^2-9}}{2x}\,dx$$

We can draw a right triangle labeled this way:



From this we can read off $2x = 3\sec\theta$, so $dx = \frac{3}{2}\sec\theta\,\tan\theta$, and $\sqrt{4x^2-9} = 3\tan\theta$. Putting those pieces into the integral we get some cancellation and are left with $\frac{3}{2}\int\tan^2\theta\,d\theta$. Using $\tan^2\theta = \sec^2\theta - 1$, we have $\frac{3}{2}\int\sec^2\theta\,d\theta - \frac{3}{2}\int 1\,d\theta = \frac{3}{2}\tan\theta - \frac{3}{2}\theta + C$. Substituting back for x's we get $\frac{1}{2}\sqrt{4x^2-9} - \frac{3}{2}\arctan\left(\frac{\sqrt{4x^2-9}}{3}\right) + C$.

(b) $$\int \cos^2(x)\sin^3(x)\,dx$$

We keep one copy of $\sin x$ with dx, and convert the remaining $\sin^2 x$ to $1-\cos^2 x$. We get $\int\cos^2 x(1-\cos^2 x)\sin x\,dx$. Multiplying out and separating we have $\int\cos^2 x\sin x\,dx - \int\cos^4 x\sin x\,dx$. In each of these integrals use the substitution $u = \cos x$, and we get $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$.

(c) $$\int x\cos(2x)\,dx$$

This seems a natural to do with integration by parts. I will start with $u = x$ and $dv = \cos(2x)\,dx$, so $du = dx$ and $v = \frac{1}{2}\sin(2x)$. Then the integral can be written as $\frac{1}{2}x\sin(2x) - \frac{1}{2}\int\sin(2x)\,dx$ $= \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$.

Problem 2 (10 points)
Evaluate the integrals:

(a) $$\int_0^3 \frac{dx}{(x-2)^{\frac{2}{3}}}$$

This is an improper integral: The denominator goes to zero at $x = 2$ which is within the range of integration. We first split it into two integrals that have the "problem" at their endpoints,

$$\int_0^2 \frac{dx}{(x-2)^{\frac{2}{3}}} + \int_2^3 \frac{dx}{(x-2)^{\frac{2}{3}}}.$$

Now we use a limit on each to hold back from the trouble spots:

$$\lim_{b\to 2^-}\int_0^b \frac{dx}{(x-2)^{\frac{2}{3}}} + \lim_{a\to 2^+}\int_2^3 \frac{dx}{(x-2)^{\frac{2}{3}}}.$$

Now each of these integral is one to which the Fundamental Theorem applies, so we can evaluate it by finding antiderivatives using the power rule. $\int\frac{dx}{(x-2)^{\frac{2}{3}}} = 3(x-2)^{\frac{1}{3}}$ so the limits become

$$\lim_{b\to 2^-}\left[3(b-2)^{\frac{1}{3}} - 3(0-2)^{\frac{1}{3}}\right] + \lim_{a\to 2^+}\left[3(3-2)^{\frac{1}{3}} - 3(a-2)^{\frac{1}{3}}\right].$$

The parts with $(b-2)$ and $(a-2)$ go to zero as a and b go to two. We are left with $-3\sqrt[3]{-2}+3\sqrt[3]{1} = 3\sqrt[3]{2}+3$.

(b) $$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,dx$$

We start with the substitution $u = \sin(x)$, so $du = \cos(x)\,dx$. The integral becomes $\int \frac{du}{1+u^2}$ but we should change the limits to go with the new variable. When $x = 0$, $u = \sin 0 = 0$, and when $x = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$. So now we have

$$\int_0^1 \frac{du}{1+u^2}.$$

You may recognize that as giving an arctangent. If you did not recognize it, you could evaluate this with a triangle and a trig substitution and get the same answer. We get $\arctan(1) - \arctan(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$.

Problem 3 (15 points)

For each of these sequences: (i) Tell whether it converges or not. (ii) If it converges, tell what its limit is. You should be sure to give reasons, but you do not need the precision of ϵs and the formal definition of the limit of a sequence.

(a) $$a_n = \sin\left(\frac{\pi}{3} + \frac{1}{n}\right)$$

Since $\sin(x)$ is a continuous function, we can find the limit of $\frac{\pi}{3} + \frac{1}{n}$ and take the *sin* of that result. In this limit $\frac{\pi}{3}$ does not change. The limit of $\frac{1}{n}$ is 0: The numerator is fixed while the denominator is growing without bound. So the limit of $\frac{\pi}{3} + \frac{1}{n}$ is $\frac{\pi}{3}$. Then we just take the sin and get $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ as the answer.

(b) $$a_n = \sin\left(\frac{\pi}{3} + n\pi\right)$$

Consider what happens for different values of n: If n is odd, e.g. $n = 1$, we have $\sin(\frac{4\pi}{3})$. (For any other odd n, the "angle" will differ from this one by a multiple of 2π which will give the same value for the sin.) You can work out what that is, but it is enough to note that the angle is in the second quadrant where the sin is positive. But for any even value of n, the angle will be exactly π away from that one, in the fourth quadrant where the sin is negative. So this sequence oscillates between two values, one positive and one negative, and has no limit.

(c) $$a_n = \left(\frac{n-1}{n}\right)^n$$

Using a little algebra, $a_n = (1-\frac{1}{n})^n$ which is the same as $(1+\frac{x}{n})^n$ if $x = -1$. Hence the sequence converges to $e^{-1} = \frac{1}{e}$.

Problem 4 (15 points)

For each of the following series: (i) Tell whether it converges. (ii) If it converges, tell what it converges to. Be sure to give reasons for your answers!

(a) $$\sum_{n=0}^{\infty}\left(\frac{4}{3^n} - \frac{1}{2^n}\right)$$

This can be treated as two separate series, each a convergent geometric series, where one is subtracted from the other.

$$\sum_{n=0}^{\infty}\left(\frac{4}{3^n} - \frac{1}{2^n}\right) = \sum_{n=0}^{\infty}\frac{4}{3^n} - \sum_{n=0}^{\infty}\frac{1}{2^n}.$$

The first is geometric with $a = 4$ and $r = \frac{1}{3}$ and so converges to $4(\frac{1}{1-\frac{1}{3}}) = 6$. The second is geometric with $a = 1$ and $r = \frac{1}{2}$ and so converges to $\frac{1}{1-\frac{1}{2}} = 2$. (We know they are convergent because in each case $|r| < 1$.) Hence the sum of the original series is $6 - 2 = 4$.

(b) $\sum_{n=0}^{\infty}\left(\left(\frac{4}{3}\right)^n - \frac{1}{2^n}\right)$

Again we look at this series as a combination of two series, but this time the first one is geometric with $r = \frac{4}{3}$ and so diverges. The second one converges and so the combination diverges. You can also note that the terms of this series get larger as n increases and do not go to zero, so by the n^{th} term test it diverges. There are other tests that would also show it diverges.

(c) $\sum_{n=1}^{\infty}\frac{n^2-4}{n}$

The terms of this series get larger without bound as $n \to \infty$, rather than going to zero. So the n^{th} term test again tells us it diverges.

Problem 5 (15 points)
For each of the following series: (i) Tell whether it converges absolutely, converges conditionally, or diverges. Be sure to give reasons for your answers!

(a) $\sum_{n=1}^{\infty}\frac{\cos(n)}{n^{\frac{3}{2}}}$

The absolute value of a_n for this series is $\frac{|\cos(n)|}{n^{\frac{3}{2}}}$ which is $\leq \frac{1}{n^{\frac{3}{2}}}$. A series with the latter terms would be a p-series with $p = \frac{3}{2} > 1$ and so would converge. Since the terms of our series, after taking absolute values, are smaller than those of a convergent series, our series converges absolutely.

(b) $\sum_{n=2}^{\infty}(-1)^n\frac{1}{\ln(n^4)}$

This time if we take the absolute value of the term a_n we get $\frac{1}{\ln(n^4)} = \frac{1}{4\ln(n)} = \frac{1}{4}\frac{1}{\ln(n)}$. Since $\ln(n) < n$, $\frac{1}{\ln(n)} > \frac{1}{n}$. Since the series $\sum\frac{1}{n}$, the harmonic series, diverges, this series of absolute values diverges by comparison. So our series does not converge absolutely. But it might converge if we do not take absolute values. We check and it satisfies the three conditions of Leibniz' theorem: The terms are alternating in sign, they are decreasing in magnitude, and their limit is zero. Hence by Leibniz' theorem the series as originally given converges, but it does not converge absolutely, so it converges conditionally.

(c) $\sum_{n=1}^{\infty}(-1)^n\frac{n}{3n+5}$

The terms of this series approach $\frac{1}{3}$ as $n \to \infty$, so they do not have limit zero. Hence by the n^{th} term test this series cannot converge, i.e. it diverges.

Problem 6 (5 points)
For the power series

$$\sum_{n=0}^{\infty}\frac{(x+3)^n}{5^n}$$

Find the radius of convergence, and also find the interval of convergence. Be sure to show your reasoning.
We apply the ratio test to the absolute values of the terms of the series.

$$\rho = \lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty}\frac{\frac{|x+3|^{n+1}}{5^{n+1}}}{\frac{|x+3|^n}{5^n}} = \lim_{n\to\infty}\frac{|x+3|}{5} = \frac{|x+3|}{5}.$$

By the ratio test the series will converge absolutely when $\rho < 1$, i.e. $\frac{|x+3|}{5} < 1$. Rearranging algebraically, $-1 < \frac{x+3}{5} < 1$ or $-5 < x+3 < 5$. Subtracting 3 we get $-8 < x < 2$. But the ratio test does not tell us what happens if $\rho = 1$ so we need to check the endpoints: In this case at both $x = -8$ and $x = 2$ the terms of the series (alternately ± 1 at $x = -8$ and consistently 1 at $x = 2$) do not go to zero so the series diverges. Hence the interval of convergence is $(-8, 2)$. This tells us that the series converges for x values up to 5 units away from the center of the interval ($x = -3$) so the radius of convergence is 5.

Problem 7 (12 points)

(a) Find the Maclaurin series for $f(x) = \ln(1-x)$.

One way to find this series is to note that the derivative of $\ln(1-x)$ is $-\frac{1}{1-x}$ and that looks like our formula for the sum of a geometric series. From that one can write out a series for $-\frac{1}{1-x}$ and integrate it term-by-term to get a series for $\ln(1-x)$.

Here, however, is the more direct way from the formulas we have for coefficients in a Maclaurin series. I start with a table of the various derivatives and their values at $x = 0$:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1-x)$	0
1	$\frac{-1}{(1-x)}$	-1
2	$\frac{-1}{(1-x)^2}$	-1
3	$\frac{-2}{(1-x)^3}$	-2
4	$\frac{-6}{(1-x)^4}$	-6

We can see that each time we take the derivative from here on: (a) the chain rule with $-x$ in the denominator will require us to multiply by -1 but also the negative power corresponding to $(1-x)$ being in the denominator will require a $-$ sign, so the signs will all stay $-$. (b) The power will go one more step negative each time, so the numbers 1, 2, 6 will continue to get multiplied each time by one higher number and produce factorials. (c) Putting in $x = 0$ will always give 1 for the denominator of the fraction. Hence the n^{th} derivative, at zero, will always be negative and will step through the factorials but one step delayed, giving $-(n-1)!$. Hence when we compute the coefficients a_n for the Maclaurin series we get $a_n = -\frac{(n-1)!}{n!} = -\frac{1}{n}$, for each $n \geq 1$. Since we had $a_0 = 0$ which does not fit that pattern it is easiest to start the sum at 1 rather than 0.

Thus the series is

$$\sum_{n=1}^{\infty} -\frac{1}{n}x^n = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots - \frac{x^n}{n} - \cdots$$

(b) For what values of x does the series converge absolutely?

The ratio of successive terms gives us

$$\left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = |x| \frac{n}{n+1}$$

and the limit as $n \to \infty$ is $\rho = |x|$. Hence the series converges absolutely when $|x| < 1$ or $-1 < x < 1$. (At $x = \pm 1$ the series of absolute values is the harmonic series which diverges.)

Problem 8 (13 points)

Suppose we use the terms of the Maclaurin series for $f(x) = e^{3x}$ through the x^3 term to compute an approximation to $e^{0.3}$. Give a bound for how far the approximation might differ from the true value. Use a calculus result to produce your bound: No credit will be given for using a calculator to give the

difference between these numbers.
You may wish to use the fact that $e < 3$. Be careful in choosing a method not to apply an alternating-series result unless the series is truly alternating!
The Maclaurin series for e^{3x} is

$$1 + 3x + \frac{(3x)^2}{2} + \frac{(3x)^3}{6} + \frac{(3x)^4}{24} + \cdots.$$

To use this to compute $e^{0.3}$ we would set $x = 0.1$ so that $3x = 0.3$. Hence we want to estimate the remainder term $R_3(0.1, 0)$ for $f(x) = e^{3x}$. In general $R_3(x, 0) = \frac{f''''(c)}{4!}x^4$. For $f(x) = e^{3x}$, the fourth derivative is $81e^{3x}$. Hence the remainder term is $\frac{81e^{3c}}{24}(0.1)^4$ where c is some number between 0 and 0.1. The largest this can be is $\frac{81e^{0}.3}{24}(0.0001)$. Since we don't know $e^{0.3}$ we use $e^{0.3} < e < 3$ and have the remainder term is less than $\frac{81 \cdot 3}{24}(0.0001) = 0.0010125$.