Mathematics 222 **Summer 2002** **Wilson**

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ANSWERS

Problem 1

(a) Find the solution to $y' + 3x^2y = e^{2x-x^3}$ that satisfies $y(0) = \dfrac{5}{2}$.

This is already in the form we used for a first order linear differential equation, with $P(x) = 3x^2$ and $Q(x) = e^{2x-x^3}$. We calculate $\int P(x)\,dx = \int 3x^2\,dx = x^3$ (ignoring the constant of integration) and have as our integrating factor $\rho = e^{x^3}$. The general solution the the differential equation will then be $y = \frac{1}{\rho}\int \rho Q(x)\,dx = e^{-x^3}\int e^{x^3}e^{2x-x^3}\,dx = e^{-x^3}\int e^{2x}\,dx$ $= e^{-x^3}(\frac{1}{2}e^{2x} + C)$.

Now we need to choose C so that this fits $y(0) = \frac{5}{2}$: Putting 0 in for x and $\frac{5}{2}$ in for y we get $\frac{5}{2} = \frac{1}{2} + C$ and so $C = 2$. Thus the solution is $y = e^{-x^3}(\frac{1}{2}e^{2x} + 2)$.

(b) What will be the value of $y(1)$?

Using the result from (a), $y(1) = e^{-1}(\frac{1}{2}e^2 + 2) = \frac{1}{2}e + 2e^{-1}$. This can be rewritten in various ways, such as $\frac{e}{2} + \frac{2}{e}$, or approximated using a calculator as $2.0949\ldots$.

Problem 2

For each of the following series, tell all you can about its convergence. This should include whether it converges or diverges, and if appropriate whether it converges absolutely or conditionally. Be sure to give reasons for your answers!

(a) $$\sum_{n=1}^{\infty}(-1)^n\left(\frac{\ln(n)}{n}\right)$$

Since this series has some negative and some positive terms, it is appropriate to consider absolute vs. conditional convergence. If we take absolute values we have $\sum \frac{\ln(n)}{n}$: We note that $\frac{\ln(n)}{n} > \frac{1}{n}$ and that $\sum \frac{1}{n}$ diverges, so the series of absolute values diverges by comparison. The terms $\frac{\ln(n)}{n}$ of the series do decrease with limit zero as $n \to \infty$, and the signs alternate, so by Leibniz' theorem the series does converge. Hence it converges conditionally.

(b) $$\sum_{n=1}^{\infty}\cos\left(\frac{4}{n}\right)$$

As $n \to \infty$, $\frac{4}{n} \to 0$, and the cosine is a continuous function, so the terms $\cos(\frac{4}{n})$ in this series converge to $\cos(0) = 1$. Hence the terms do not go to zero so the series cannot converge. (The "n^{th} term test".)

(c) $$\sum_{n=1}^{\infty}\frac{3}{5^n}$$

This series converges, and there are many ways to show it does. The most direct may be to note that it is a geometric series whose first term is $\frac{3}{5}$, with ratio $r = \frac{1}{5}$, and that a geometric series with $|r| < 1$ converges.

(d) $$\sum_{n=1}^{\infty}(-1)^n\left(\frac{2}{n\sqrt{n}}\right)$$

Since there are negative terms we first test for absolute convergence: $\sum \frac{2}{n\sqrt{n}} = 2\sum \frac{1}{n^{\frac{3}{2}}}$ is a p-series with $p = \frac{3}{2} > 1$ and so it converges. Hence the series converges absolutely.

Problem 3

Consider the curve given parametrically as $x = -2 - \sin t$ and $y = 3 + 3\cos t$ for $0 \le t \le 2\pi$.

(a) Describe the path this traverses: Where does it start (when $t = 0$), how does it progress and along what path, and where does it end? (The path is one of the conic sections we have studied. You should identify it as such.)

(b) Give an equation in rectangular (Cartesian) form relating x and y without the parameter t. (You may find the identity $\sin^2 t + \cos^2 t = 1$ useful in deriving this equation.)

If we write $x + 2 = -\sin t$ and $y - 3 = 3\cos t$, we have $(x+2)^2 + \frac{(y-3)^2}{9} = 1$. Hence the path is an ellipse. Its center is at $(-2, 3)$ and it extends horizontally from $(-3, 3)$ to $(-1, 3)$ and vertically from $(-2, 0)$ to $(-2, 6)$.
When $t = 0$ we have $x = -2 - \sin(0) = -2$ and $y = 3 + 3\cos(0) = 6$ so we start at $(-2, 6)$, the top of the ellipse. As t increases, $\sin t$ increases and $\cos t$ decreases so x decreases and y also decreases: We move counter-clockwise around the ellipse, begining in a "southwesterly" direction. When t reaches 2π the values of x and y have returned to their original values, so we return to the starting point $(-2, 6)$.

Problem 4
Find the area inside <u>one leaf</u> of the three-leafed rose $r = \sin(3\theta)$.
To determine what range of θ values consititute "one leaf" we solve $r = 0$, $\sin(3\theta) = 0$. We get that 3θ must be a multiple of π. The first two such are $3\theta = 0$ and $3\theta = \pi$, i.e. $\theta = 0$ and $\theta = \frac{\pi}{3}$, and I will calculate the area in that leaf. We get

$$\int_0^{\frac{\pi}{3}} \frac{1}{2}(\sin 3\theta)^2\, d\theta = \frac{1}{4}\int_0^{\frac{\pi}{3}} (1 - \cos 6\theta)\, d\theta = \frac{1}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\frac{\pi}{3}} = \frac{1}{4}\left(\frac{\pi}{3} - \frac{1}{6}\sin 2\pi + \frac{1}{6}\sin 0\right) = \frac{\pi}{12}.$$

Problem 5
Use words to describe the following sets of points in space.

(a) $y = 0$ and $z \ge |x|$

The restriction $y = 0$ implies the points must be in the plane that contains the x and z axes. If we had $z = |x|$, that would correspond to a "V" of two rays extending out from the origin, tilted up 45 degrees, within the plane $y = 0$. Since we have $z \ge |x|$ all the points in the $y = 0$ plane above that V are also included. Hence the set is the sheet in the x-z plane coming down to a point at the origin and bounded on the bottom by lines tilted up at 45 degrees.

(b) $4 \le x^2 + y^2 + z^2 \le 9$

Since $x^2 + y^2 + z^2$ is the square of the distance from the origin to the point (x, y, z), this set consists of the points whose distance from the origin is between 2 and 3. Hence it is a "hollow ball", centered at the origin: The inside is the sphere of radius 2 centered at the origin, the outside is the sphere of radius 3 centered at the origin, and the set contains all points on those spheres as well as the points between them.

(c) $\frac{\pi}{2} \le \phi \le \pi$ and $0 \le \theta \le \frac{\pi}{2}$.

This is apparently a set of spherical coordinates. The inequalities $\frac{\pi}{2} \le \phi \le \pi$ restrict the points to lying on or below the "floor", the x-y plane. The inequalities $0 \le \theta \le \frac{\pi}{2}$ restrict the points to lying between the plane where $y = 0$ and the plane where $x = 0$. There is no restriction on the distance ρ from the origin. So the set is one eighth of space: It consists of the points which are "southeast" of the origin, in terms of their x-y location, but at or below the floor level, i.e. where $z \le 0$.

(d) $r = z^2$

The distance r outward from the z axis is to be equal to z^2. That is distance outward in any direction, since there is no restriction on θ. If we look at a cross-section obtained by cutting down along the z-axis, i.e. in any vertical plane that goes through the origin, we will see two parabolas. They are $r = z^2$ going outward away from the z-axis in both directions in our plane. Since that is true for any such plane, the set is the surface obtained by rotating these parabolas around the z-axis. It is sort of an infinitely large "pulley", with a groove going around the z-axis.

Problem 6
Let $\vec{v} = \vec{i} + 2\vec{j} - \vec{k}$ and $\vec{u} = 8\vec{i} + 4\vec{j} - 12\vec{k}$.

(a) Find the (scalar) component of $\vec{u}$ in the direction of $\vec{v}$.

We can compute this as $\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \frac{8+8+12}{\sqrt{1+4+1}} = \frac{28}{\sqrt{6}}$.

(b) Find the (vector) projection of $\vec{u}$ on $\vec{v}$.

We could use the result from (a) and divide by the length of $\vec{v}$ and multiply the resulting number onto $\vec{v}$. Or we can start from scratch with the formula $\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{28}{6}\vec{v} = \frac{14}{3}\vec{i} + \frac{28}{3}\vec{j} - \frac{14}{3}\vec{k}$.

(c) Write $\vec{u}$ as a sum of two vectors, one parallel to $\vec{v}$ and one orthogonal to $\vec{v}$.

The vector we use parallel to $\vec{v}$ is $Proj_{\vec{v}}\vec{u}$ which we computed in (b) to be $14(\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{1}{3}\vec{k})$. If we call this $\vec{u}_1$, we now let $\vec{u}_2 = \vec{u} - \vec{u}_1$ be the other vector: This gives us $\vec{u}_2 = (8 - \frac{14}{3})\vec{i} + (4 - \frac{28}{3})\vec{j} + (-12 + \frac{14}{3}\vec{k}) = \frac{10}{3}\vec{i} - \frac{16}{3}\vec{j} - \frac{22}{3}\vec{k}$.

Problem 7
Let $A = (1, 2, -1)$, $B = (2, 2, 0)$, and $C = (1, 3, 1)$ be three points in space in ordinary rectangular (Cartesian) coordinates.

(a) What is the area of the triangle whose vertices are A, B, and C?

We need two vectors lying along sides of the triangle: I will use $\overrightarrow{AB} = \vec{i} + 0\vec{j} + \vec{k}$ and $\overrightarrow{AC} = 0\vec{i} + \vec{j} + 2\vec{k}$. The area we want is then $\frac{1}{2}$ of the size of the cross product of the vectors. Computing, $\overrightarrow{AB} \times \overrightarrow{AC} = -\vec{i} - 2\vec{j} + \vec{k}$. The length of this vector is $\sqrt{1+4+1} = \sqrt{6}$ so the area of the triangle is $\frac{\sqrt{6}}{2}$.

(b) Find a vector of unit length perpendicular to the plane containing these three points.

One vector perpendicular to the plane is the cross product we just computed, so we find a unit vector in that direction. We have already computed the length of that vector and we get for the desired vector $\frac{1}{\sqrt{6}}(-\vec{i} - 2\vec{j} + \vec{k}) = \frac{-1}{\sqrt{6}}\vec{i} + \frac{-2}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}$.

(c) Find an equation for the plane containing these three points.

We could use either the cross product computed in (a) or the unit vector from (b) as a vector perpendicular to the plane. Avoiding the square roots, I will use $-\vec{i} - 2\vec{j} + \vec{k}$. The plane goes through A, B, and C so we can use any of these as "a point on the plane." If we use A we get for the plane we get $(x-1)(-1) + (y-2)(-2) + (z+1)(1) = 0$ or $-x - 2y + z = -1 - 4 - 1 = -6$. This is a little prettier as $x + 2y - z = 6$. We can check that each of the three points A, B, and C satisfies this equation.

(d) Find equations for the line through B perpendicular to the plane containing the three points.

We can use the same vector as the direction for this line. We want then the line through $(2, 2, 0)$ in the direction of $-\vec{i} - 2\vec{j} + \vec{k}$. In parametric form this is $x = 2 - t$, $y = 2 - 2t$, $z = t$. In symmetric or Cartesian form it is $\frac{x-2}{-1} = \frac{y-2}{-2} = \frac{z}{1}$.

Problem 8

Consider the graph of

$$9x^2 - 16y^2 - 36x - 128y - 364 = 0.$$

(a) For some translated coordinates x' and y', the center (or vertex if this is a parabola) will be at the origin of the coordinate system. Write the equation in x' and y'.

We complete the squares and get $9(x-2)^2 - 16(y+4)^2 = 144$. If we let $x' = x - 2$ and $y' = y + 4$, and divide by 144, we get

$$\frac{(x')^2}{16} - \frac{(y')^2}{9} = 1.$$

(b) Is this graph a circle, an ellipse, a parabola, or a hyperbola?

Since we have it in standard form and the $(x')^2$ and $(y')^2$ terms have opposite signs, it is a hyperbola.

(c) If it is a circle, ellipse, or hyperbola, where is its center? If it is a parabola, where is its vertex? (Be sure to use (x, y) coordinates for this and the following answers!)

The center is where $x' = 0$ and $y' = 0$, so $x = 2$ and $y = -4$. The center is $(2, -4)$.

(d) (i) If it is a circle, what is the radius? What are the coordinates of two points on the circle?

(ii) If it is an ellipse, where are the ends of the major axis (the two points furthest from the center)? Where are the foci? What is the eccentricity?

(iii) If it is a parabola, where is the focus? Give an equation for the directrix.

(iv) If it is a hyperbola, where are the foci? Where does it cross the x' or y' axes (the translated axes you found in (a)). (You can identify these points using either coordinates but make clear which you are using.) Give equations (in x and y) for the asymptotes.

Using the notation we have used in class and in the text, $a = 4$ and $b = 3$. Hence $c = \sqrt{a^2 + b^2} = 5$. Thus the foci are 5 units from the center, along the x' axis and so horizontally displaced from the center. That puts them at $(-3, -4)$ and $(7, -4)$. The curve does not cross the y' axis since setting $x' = 0$ gives $(y')^2 = -9$. It crosses the x' axis at $x' = \pm a = \pm 4$. These crossings are at $(\pm 4, 0)$ in x'-y' coordinates, or $(-2, -4)$ and $(6, -4)$ in x-y coordinates. The asymptotes are $y' = \pm\frac{3}{4}x'$: Using $x' = x - 2$ and $y' = y + 4$ we get $y + 4 = \frac{3}{4}(x - 2)$ and $y + 4 = -\frac{3}{4}(x - 2)$. Those can be simplified to $y = \frac{3}{4}x - \frac{11}{2}$ and $y = -\frac{3}{4}x - \frac{5}{2}$.

Problem 9

Some initial terms of the Maclaurin series for e^{-x} are used to approximate $f(x) = e^{-x}$, for $-0.1 < x < 0.1$. How many terms are needed in order to guarantee that the approximation is within ± 0.0001 of the actual value of $f(x)$ for all x in that range? Write out the polynomial showing those terms.

Your answer should both tell the number of terms needed (or the degree of the highest power term needed, but be sure to indicate which!) and the polynomial you would use as your approximation.

(You may use the fact that $|e^x|$ and $|e^{-x}|$ are both less than 2 for $-0.1 < x < 0.1$.)

The Maclaurin series for e^{-x} starts out $1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \ldots$. The derivatives of e^{-x} are alternately e^{-x} and $-e^{-x}$. Hence the remainder term $R_n(x, 0)$ (in Lagrange form) is $\frac{\pm e^{-c} x^{n+1}}{(n+1)!}$. Thus if we use the terms of the series through $\pm\frac{x^n}{n!}$ we are guaranteed the approximation at x is within $|\frac{\pm e^{-c} x^{n+1}}{(n+1)!}|$ of the correct value, for some c between 0 and x. We need to choose n so that, for any x between ± 0.1, that is guaranteed to be at most 0.0001. Since $-0.1 < x < 0.1$ and c is between 0 and x, we know $-0.1 < c < 0.1$ also. Using the hint, $|e^{-c}| < 2$. Since $|x^{n+1}|$ increases as we move away from 0, the largest that part can be is $(0.1)^{n+1}$. Hence the

remainder term satisfies $|R_n(x,0)| \leq \frac{2\times(0.1)^{n+1}}{(n+1)!}$. We compute this for a few values of n: When $n=2$ we get $\frac{2\times 0.001}{6} = 0.00033\ldots$ which is not within the specified tolerance of 0.0001. When $n=3$ we get $\frac{2\times 0.0001}{24} = \frac{0.0001}{12} < 0.0001$ so $n=3$ is good enough.
Summarizing, we use the terms through $-\frac{x^3}{6}$, the first four terms including the constant, so we use the polynomial $1 - x + \frac{x^2}{2} - \frac{x^3}{6}$.

Problem 10

Find all solutions of $y'' + 2y' = 4x + 3$.

(a) Find all solutions of $y'' + 2y' = 0$:

The characteristic equation is $r^2 + 2r = 0$, with roots $r = -2$ and $r = 0$, so the solutions to $y'' + 2y' = 0$ are $y_h = C_1e^{-2x} + C_2e^{0x} = C_1e^{-2x} + C_2$.

(b) Find a solution of $y'' + 2y' = 4x + 3$:

Since the function $4x + 3$ is a polynomial, and 0 is a root of the characteristic equation, we use a polynomial $Ex^2 + Fx$ for our particular solution y_p to $y'' + 2y' = 4x + 3$. Then $y_p' = 2Ex + F$ and $y_p'' = 2E$. Putting these in the equation we get $2E + 4Ex + 2F = 4x + 3$. Rearranging we have $(4E)x + (2E + 2F) = 4x + 3$, so $4E = 4$ and $2E + 2F = 3$. These give us $E = 1$ and $F = \frac{1}{2}$. Hence our solution is $y_p = x^2 + \frac{x}{2}$.

(c) Write an expression for all solutions of $y'' + 2y' = 4x + 3$:

We combine the functions y_h and y_p and get $C_1e^{-2x} + C_2 + x^2 + \frac{x}{2}$ representing all solutions to the full equation.