Mathematics 222, Spring 2008 Lecture 1 (Wilson)

Final Exam May 14, 2008 ANSWERS

Problem 1

Evaluate the integrals:

(a) $\int x^2 \ln(x) dx$

ANSWER: This looks to be a candidate for integration by parts: We see a product, and if we take the derivative of $\ln x$ what we get is in some ways simpler. So we let $u = \ln(x)$ and $dv = x^2$, making $du = \frac{1}{x}dx$ and $v = \frac{x^3}{3}$. Hence $\int x^2 \ln(x)dx = \dfrac{1}{3}x^3 \ln(x) - \int \dfrac{x^3}{3}\dfrac{1}{x}dx$ $= \frac{1}{3}x^3 \ln(x) - \frac{1}{3}\int x^2\, dx = \frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + C$.

(b) $\int_0^{\frac{\pi}{3}} \sin^3(x)\, dx$

ANSWER: We can separate $\sin^3(x)$ as the product of $\sin(x)$ and $\sin^2(x)$ and use $\sin^2 x + \cos^2 x = 1$ to convert the second part to powers of $\cos x$, with $\sin x$ ready to use for du in a substitution. We have $\int_0^{\frac{\pi}{3}} \sin^3(x)\, dx = \int_0^{\frac{\pi}{3}} (\sin^2(x))\sin(x)\, dx = \int_0^{\frac{\pi}{3}} (1-\cos^2(x))\sin(x)dx$ $= \int_0^{\frac{\pi}{3}} \sin(x)dx - \int_0^{\frac{\pi}{3}} \cos^2(x)\sin(x)dx$. Using the substitution $u = \cos x$ on the second integral we get $\left[-\cos(x) + \frac{1}{3}\cos^3 x\right]_0^{\frac{\pi}{3}} = (-\frac{1}{2} + \frac{1}{24}) - (-1 + \frac{1}{3}) = \frac{5}{24}$.

(c) $\int_0^1 \dfrac{dx}{\sqrt{1-x^2}}$

ANSWER: This is an improper integral: The denominator goes to zero at $x = 1$. So we have to rewrite it as a limit. $\int_0^1 \dfrac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-} \int_0^b \dfrac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-} [\arcsin x]_0^b =$ $\lim_{b\to 1^-} (\arcsin(b) - \arcsin(0)) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$.

Problem 2

(a) An object moves along the parabola $x = -4y^2$. At the instant when $t = 0$ it is at the point $(-16, 2)$. It moves through the origin, moving along the curve without backing up, and reaches the point $(-36, -3)$ at the instant when $t = 5$. (The second point was printed as $(-36, 3)$ on the exam and corrected in the exam room.)

Find a parametric representation for this motion. I.e., find functions $f(t)$ and $g(t)$ such that setting $x = f(t)$ and $y = g(t)$ for $0 \le t \le 5$ gives position (x, y) as described above.

ANSWER: We are supposed to have t go from 0 to 5. y also changes by 5 units, from 2 to -3, which suggests letting $y = 2 - t$. Then we use $x = -4y^2$ to get $x = -4(2-t)^2$. This is enough of an answer, but you can write out in more detail that $f(t) = -4(2-t)^2 = -16 + 16t - 4t^2$ and $g(t) = 2 - t$. You can also check that $(f(0), g(0)) = (-16, 2)$ and $(f(5), g(5)) = (-36, -3)$ as required.

(b) A conic section has the following properties:

- It crosses the $y-$axis at $(0, \pm 3)$.
- It has foci at $(0, \pm 5)$.
- It is symmetric about the $y-$axis and also about the $x-$axis.
- It does not cross the $x-$axis at all.

What kind of conic section (circle, ellipse, parabola, hyperbola) is this? What is its eccentricity? Write an equation for this curve.

ANSWER: A conic section symmetric about both axes but only intersecting one of them would have to be an hyperbola. From the symmetry and the fact that the foci are on the $y-$axis we know it must have an equation of the form $-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$. The points where $x = 0$ have $y = \pm a$ and we are told the curve crosses the $y-$axis at $y = \pm 3$, so $a = 3$. The foci will be at $(0, \pm c)$, where $c^2 = a^2 + b^2$, so $c = 5$. Hence $b^2 = c^2 - a^2 = 25 - 9 = 16$ and $b = 4$. So now we know the equation is $-\frac{x^2}{16} + \frac{y^2}{9} = 1$. The eccentricity is $\frac{c}{a} = \frac{5}{3}$. (We note that is bigger than 1, appropriate for a hyperbola.)

Problem 3

(a) Let $f(x) = \ln(1 + x)$. Find the Maclaurin series for $f(x)$ through the term with x^3. Be sure to show how the coefficients are found, using derivatives and factorials: Do not just recite the series from memory or your notes.

ANSWER: The zero-th through third derivatives of $f(x)$ are $\ln(1+x)$, $\frac{1}{1+x}$, $-\frac{1}{(1+x)^2)}$, and $\frac{2}{(1+x)^3}$. Evaluating those at $x = 0$ we get $\ln(1 + 0) = \ln(1) = 0$, $\frac{1}{1+0} = 1$, $-\frac{1}{(1+0)^2} = -1$, and $\frac{2}{(1+0)^3} = 2$. Hence the first four coefficients are $a_0 = 0$, $a_1 = \frac{1}{1!} = 1$, $a_2 = \frac{-1}{2!} = -\frac{1}{2}$, and $a_3 = \frac{2}{3!} = \frac{1}{3}$. So the first terms of the series are $0 + x - \frac{x^2}{2} + \frac{x^3}{3}$.

(b) Let $p(x)$ be the polynomial $p(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$. Suppose we use this (some of the initial terms of the Maclaurin series for e^x) to approximate $e = e^1$. How accurate will this approximation be?

ANSWER: These are the terms through the fifth degree term, so I will use $R_5(x)$ in bounding the error. Since the n^{th}-derivative of e^x is e^x for any positive integer n, we have $R_5(x) = \frac{e^c x^6}{6!}$ for some c between 0 and x. We are evaluating $p(1)$ so $x = 1$, hence $x^6 = 1$. c is some number between 0 and 1: e^x is an increasing function so the largest e^c can be will be $e^1 = e$. Now we have a problem, we want to evaluate e but we need it to bound the error. But we know at least that $e \leq 3$, so I will use $e^c \leq 3$. Thus $R_5(1) \leq \frac{3 \times 1}{6!}$ $= \frac{3}{720} = \frac{1}{240}$, so the polynomial approximation is off by no more than $\frac{1}{240}$.

Note that $\frac{1}{240} \approx 0.0041666$ while the difference between a calculator value for e and for $p(1)$ is about 0.00161516, so the actual error seems comfortably inside our bound but not so far that we probably could have found a much better bound.

Problem 4

Find the general solution $y(x)$ to $y'' - 4y' + 4y = 2\sin(2x)$.

ANSWER: First we find the roots of the characteristic equation $r^2 - 4r = 4 = 0$, or $(r-2)^2 = 0$. We see that $r = 2$ is a double root. So the general solution to the homogeneous equation $y'' - 4y' + 4y = 0$ is $y(x) = c_1 e^{2x} + c_2 x e^{2x}$. Now to find a particular solution of the original equation, using the fact that $2i$ is not a root of the characteristic equation (although 2 is), we try $y_p = A\cos 2x + B\sin 2x$. Then $y_p' = -2A\sin 2x + 2B\cos 2x$ and $y_p'' = -4A\cos 2x - 4B\sin 2x$. Putting those into the equation, $-4A\cos 2x - 4B\sin 2x + 8A\sin 2x - 8B\cos 2x + 4A\cos 2x + 4B\sin 2x = 2\sin 2x$. Collecting terms, $-8B\cos 2x + 8A\sin 2x = 2\sin 2x$, so $8B = 0$ and $8A = 2$. Thus $B = 0$ and $A = \frac{1}{4}$. Hence $y_p(x) = \frac{1}{4}\cos 2x$. So the general solution, a combination of y_h and y_p, is $y(x) = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{4}\cos 2x$.

Problem 5

Let P be the point $(1,2,5)$ and Q the point $(1,1,4)$ in space. Consider the plane $2x - 2y + z = 4$.

(a) Find the distance from the point P to the plane $2x - 2y + z = 4$.

ANSWER: If we let $\vec{n} = 2\vec{\imath} - 2\vec{\jmath} + \vec{k}$, this will make $\vec{n}$ a vector normal (orthogonal, perpendicular) to the plane. The point Q has coordinates $(1,1,4)$ satisfying the equation $2 \times 1 - 2 \times 1 + 1 \times 4 = 4$ so Q is on the plane. Hence the absolute value of the scalar projection of $\overrightarrow{PQ}$ onto $\vec{n}$ will give the distance from P to the plane. We can write $\overrightarrow{PQ}$ as $(1-1)\vec{\imath} + (1-2)\vec{\jmath} + (4-5)\vec{k} = -\vec{\jmath} - \vec{k}$. The scalar projection of that onto $\vec{n}$ will be $\frac{\overrightarrow{PQ} \bullet \vec{n}}{|\vec{n}|} = \frac{0+2-1}{\sqrt{4+4+1}} = \frac{1}{3}$. So the distance is $\frac{1}{3}$ units.

(Another method: The line through P in the direction of $\vec{n}$ has parametric equations $x = 1 + 2t$, $y = 2 = 2t$, $z = 5 + t$. If you put those into the plane equation you find the line hits the plane where $t = \frac{1}{9}$, hence at the point $(1 + \frac{2}{9}, 2 - \frac{2}{9}, 5 + \frac{1}{9})$. If you compute the distance from that point to P you also get $\frac{1}{3}$.)

(b) Write $\overrightarrow{PQ}$ as the sum of two vectors $\overrightarrow{PQ} = \vec{u} + \vec{v}$ such that $\vec{u}$ is perpendicular to the plane $2x - 2y + z = 4$ while $\vec{v}$ is perpendicular to $\vec{u}$.

ANSWER: The scalar projection $\frac{1}{3}$ computed above will be the magnitude of a vector we can use for $\vec{u}$, and the direction must be that of $\vec{n}$. So we find a unit vector in the direction of $\vec{n}$, $\frac{1}{3}\vec{n} = \frac{2}{3}\vec{\imath} - \frac{2}{3}\vec{\jmath} + \frac{1}{3}\vec{k}$, and multiply that by the desired magnitude $\frac{1}{3}$ to get $\vec{u} = \frac{2}{9}\vec{\imath} - \frac{2}{9}\vec{\jmath} + \frac{1}{9}\vec{k}$. Now if we are to get $\overrightarrow{PQ} = \vec{u} + \vec{v}$ we must have $\vec{v} = \overrightarrow{PQ} - \vec{u}$ $= (-\vec{\jmath} - \vec{k}) - (\frac{2}{9}\vec{\imath} - \frac{2}{9}\vec{\jmath} + \frac{1}{9}\vec{k}) = -\frac{2}{9}\vec{\imath} - \frac{7}{9}\vec{\jmath} - \frac{10}{9}\vec{k}$. That has to be the answer for $\vec{v}$, but we can check whether it is $\perp$ to $\vec{u}$ by computing their dot product: $\vec{u} \bullet \vec{v} = (\frac{2}{9}\vec{\imath} - \frac{2}{9}\vec{\jmath} + \frac{1}{9}\vec{k}) \bullet (-\frac{2}{9}\vec{\imath} - \frac{7}{9}\vec{\jmath} - \frac{10}{9}\vec{k}) = -\frac{4}{81} + \frac{14}{81} - \frac{10}{81} = \vec{0}$ so indeed $\vec{u} \perp \vec{v}$.

Problem 6

Let P, Q, and R be the points $P = (2,1,-3)$, $Q = (3,1,-2)$, and $R = (2,3,-1)$ in space.

(a) Find an equation for the plane passing through P, Q, and R.

ANSWER: We can find this equation easily if we know a point on the plane and a vector normal to it. We could use any of the three given points for the point on the plane. To find a vector perpendicular to the plane we construct the vector $\overrightarrow{PQ} = \vec{\imath} + \vec{k}$ and the vector $\overrightarrow{PR} = 2\vec{\jmath} + 2\vec{k}$: These are not parallel, and they each must be $\perp$ any vector perpendicular to the plane, so if we compute their cross product it will give a vector in the correct direction to use as a normal vector. $\overrightarrow{PQ} \times \overrightarrow{PR} = (\vec{\imath} + \vec{k}) \times (2\vec{\jmath} + 2\vec{k})$ $= (0 \cdot 2 - 2 \cdot 1)\vec{\imath} - (1 \cdot 2 - 1 \cdot 0)\vec{\jmath} + (1 \cdot 2 - 0 \cdot 0)\vec{k}) = -2\vec{\imath} - 2\vec{\jmath} + 2\vec{k}$. We use that as $\vec{n}$, a vector normal to the plane, and the equation for the plane can be written (using P as the point on the plane) as $-2(x-2) - 2(y-1) + 2(z-(-3)) = 0$. That can be rewritten in various prettier ways, e.g. $2x + 2y - 2z = 12$ or $x + y - z = 6$. You can check that each of the three points P, Q, and R do fit these equations.

(b) What is the area of the triangle whose vertices are P, Q, and R?

ANSWER: The area of the triangle will be $\frac{1}{2}$ of the magnitude of the cross product $\vec{n}$ computed above, $A = \frac{1}{2}|-2\vec{\imath} - 2\vec{\jmath} + 2\vec{k}| = \frac{1}{2}\sqrt{4+4+4} = \sqrt{3}$.

Problem 7

Solve the initial value problem $y' + 2xy = x$ with $y(0) = 0$.

ANSWER: This is a first order linear equation already in our standard form, i.e. if we let $P(x) = 2x$ and $Q(x) = x$ then the machinery we developed applies directly. $\int P(x)dx = \int 2x\,dx = x^2(+C)$ so $e^{\int P(x)dx} = e^{x^2}$. We could either write the equation multiplied by that integrating factor and use the fact that the left side will now be the derivative of a product or continue with the machinery. $y(x) = \frac{1}{e^{(x^2)}} \int e^{(x^2)} x dx$. We use the substitution $u = x^2$ and $du = 2x\,dx$ on the remaining integral and get $y(x) = \frac{1}{2}e^{-x^2}\left(e^{(x^2)} + C\right) = \frac{1}{2} + Ce^{-x^2}$. (You could keep $\frac{1}{2}$ in front of C: so long as you are careful with your arithmetic the end result will be the same, but C represents all real numbers and so does $\frac{1}{2}C$.)

Now we need to account for the initital condition $y(0) = 0$. The function we have so far gives $y(0) = \frac{1}{2} + Ce^0 = \frac{1}{2} + C$, so $\frac{1}{2} + C = 0$ and $C = -\frac{1}{2}$. Thus the solution to the initial value problem is $y(x) = \frac{1}{2} - \frac{1}{2}e^{-x^2}$.

Problem 8

For each of the following series, tell which (one or more) of these terms applies: (i) Diverges (ii) Converges (iii) Converges Conditionally (iv) Converges Absolutely. Be sure to give reasons!

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{n!\,n!}$

ANSWER: Those factorials stand a good chance of cancelling so I'll try the ratio test. The n^{th} term is $\frac{(2n)!}{n!n!}$ so the $(n+1)^{st}$ term is $\frac{(2n+2)!}{(n+1)!(n+1)!}$. Dividing $\frac{a_{n+1}}{a_n}$, after some algebraic rearrangement, gives $\frac{(2n+2)!}{(2n)!}\frac{n!}{(n+1)!}\frac{n!}{(n+1)!} = [(2n+2)(2n+1)]\left[\frac{1}{n+1}\right]\left[\frac{1}{n+1}\right] = \frac{4n^2+3n+2}{n^2+2n+1}$. The limit of that ratio is $\rho = 4 > 1$ so the series diverges by the ratio test. Since it diverges, it cannot converge in any way, so none of the other terms apply.

(You can also show the terms are not going to zero, so the series must diverge by the "n^{th}−term test", but I think that is harder.)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

ANSWER: This is an alternating series. The magnitude of the terms is decreasing to zero, so by Leibniz' theorem the series does converge as it is. If we take the absolute values of the terms we get $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$. We can either compare this to the harmonic series (these terms are larger) or view it as a p-series with $p = \frac{1}{2} < 1$ and either way see that it diverges. So the series converges but not absolutely, i.e. it converges conditionally. It does not diverge.

(c) $\displaystyle\sum_{n=0}^{\infty} x^n$, if $-1 < x < 1$

ANSWER: You can either use the ratio test or think of this as a geometric series with ratio $r = x$. It converges absolutely for $|x| < 1$ which is the set of x-values we are given. So the series converges absolutely, it converges, it does not converge conditionally and it does not diverge.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4+1}$

ANSWER: We start by testing for absolute convergence, i.e. we make all the terms positive: Since $n^4 + 1 > n^4$, $\frac{1}{n^4+1} < \frac{1}{n^4}$. You can show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges in several ways: It is a p-series with $p = 4 > 1$, or you can use the integral test, for example. So after making the terms of the original series positive we have a series whose terms are each less than corresponding terms of a converging series, so that series of absolute values converges. Hence the original series converges absolutely, which implies it converges. But it does not converge conditionally and it does not diverge.