Mathematics 222, Spring 2008 — Lecture 1 (Wilson)

Second Midterm Exam — April 24, 2008 — ANSWERS

Problem 1

For any given number x the formula $a_n = x^n$ for $n = 1,\ 2, \ldots$ defines numbers $a_1,\ a_2, \ldots$ Using that formula:

(a) For what choices of x does the sequence $\{a_n\}$ converge?
When it converges, to what does it converge?
Explain your reasoning.

ANSWER:

The sequence $a_n = x^n$ is one of the ones used as a specific example in the text (page 754), if $|x| < 1$. In that case the sequence does converge, to 0. But you might not remember that piece of Theorem 5, so here is a self-contained treatment. Consider five cases: (i) $|x| < 1$; (ii) $x = 1$; (iii) $x = -1$; (iv) $x > 1$; (v) $x < -1$. Any real number does fit in one of those cases.

Case (i): One way to show that this converges, and converges to zero, is to refer to the series $\sum_{0}^{\infty} x^n$ which is a geometric series with ratio x, so it converges if $|x| < 1$, and it could not converge unless its terms had limit zero. Or, you could be very formal with the ϵ definition of the limit and show how to find an N such that for all terms a_n with $n > N$, $|a_n - 0| < \epsilon$. But we would accept here that higher powers of a number smaller (in size) than 1 "clearly" get smaller and smaller, going to zero.

Case (ii): In this case the value of $a_n = 1^n$ is 1 for every n. Hence the sequence is a constant sequence, so it converges with limit equal to that constant, i.e. it converges to 1.

Case (iii): Now the terms a_n take on the values $-1,\ 1,\ -1,\ 1,$ alternately. They do not "go to infinity", but they also do not settle down to be nearer and nearer to any specific limit. This case does not converge.

Case (iv): If $x > 1$, the successive values $a_n = x^n$ will get larger and larger without a bound. So this sequence does not have a limit.

Case (v): If $x < -1$, the values get bigger and bigger in size but switch back and forth between positive and negative. So the sequence does not have a limit.

Summarizing: The sequence has a limit (0) if $|x| < 1$, a different limit (1) if $x = 1$, and diverges for all other values of x.

(b) For what choices of x does the series $\sum_{n=1}^{\infty} a_n$ converge?
When it converges, to what does it converge?
Explain your reasoning.

ANSWER:

This is a geometric series with ratio x, and with the first term $1 \neq 0$, so it converges if and only if $|x| < 1$. For those values $|x| < 1$, it converges to $\frac{1}{1-x}$.

Problem 2

Solve the initial value problem: $$e^x \frac{dy}{dx} + e^x y = \sin x \qquad \text{AND} \qquad y(0) = 0.$$

ANSWER:
If you look closely you see that the left side of the equation is exactly the derivative of $e^x y$, so we could just integrate both sides and get $e^x y = \int \sin x \, dx = -\cos(x) + C$, and then multiply by e^{-x}. But in case you don't happen to notice that, we can make this fit the form we used for first-order linear equations. Divide by e^x, or equivalently multiply by e^{-x}, to get $\frac{dy}{dx} + y = e^{-x} \sin x$. This is our form with $P(x) = 1$ and $Q(x) = e^{-x} \sin x$. We compute $\int P(x)dx$ and get x ($+C$ which we ignore) and then the integrating factor is $v(x) = e^x$. (And multiplying that back on just gives the original equation, which is why the simpler way I gave first works.) So we can write the solution as $y = \frac{1}{e^x} \int e^x e^{-x} \sin(x) \, dx = e^{-x} \int \sin(x) \, dx = e^{-x}(-\cos(x) + C)$. (Exactly the same as the first version...)
So now we need to find what value of C makes this fit the initial conditions: From this solution we have $y(0) = 1 \times (-\cos 0 + C) = -1 + C$, and the initial condition requires $y(0) = 0$, so $-1 + C = 0$ and hence $C = 1$. Thus the solution is $y = e^{-x}(-\cos(x) + 1)$.

Problem 3
For each series, tell whether it converges Absolutely, Conditionally, or Not at All: Be sure to give reasons for your answers, citing the convergence tests you used. Make sure you point out how the requirements of a test are satisfied.

(a) $\displaystyle\sum_{n=2}^{\infty} \left[(-1)^{n+1} \, \frac{\sin(2n) + 2}{n - 1} \right]$

ANSWER:

The original problem on the exam just asked whether this converges absolutely, conditionally, or not at all. If we do that version:

For any n, $\sin 2n$ is between ± 1. So the numerator is between 1 and 3. So, if we take absolute values, we get terms that are at least $\frac{1}{n-1} > \frac{1}{n}$, so by comparison to the harmonic series the resulting series diverges. So we know the series does not converge absolutely.

Now does the actual alternating series converge? This is actually a hard series to be completely precise about. The alternating series test (Leibniz' theorem) seems to apply: This is definitely an alternating series, and the limit of the terms is definitely zero. But the test also requires that eventually the terms are non-increasing in absolute value! And since $\sin 2n$ varies up and down, we can't be sure (in fact it is false) that these terms ever do become non-increasing. In fact I know it converges, to approximately -1.04, but using arguments too esoteric to expect in Math 222!

So we announced at the exam that you should assume the series does converge and just answer whether it converges absolutely or conditionally: If it is known to converge then it must do one or the other. As shown above, it does not converge absolutely, so it must converge conditionally.

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{2}{n^{\sqrt{2}}}$

ANSWER:

If we take absolute values we get a series with terms $\frac{2}{n^{\sqrt{2}}}$. That is $2\sum \frac{1}{n^{\sqrt{2}}}$, twice a p-series with $p = \sqrt{2} > 1$, so it converges.

Hence the series converges absolutely.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

ANSWER: The short answer here is that this is the harmonic series which we have been using as a standard example of a divergent series. So it diverges. The terms are positive so taking absolute values makes no difference, so conditional convergence is not possible. So this does not converge at all. (If you did not recognize the harmonic series you could view it as a p-series with $p = 1$, or apply the integral test.)

Problem 4

(a) Let $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\ \frac{x^n}{n} + \cdots$, wherever that series converges.

(i) Find the interval of convergence for this power series. Be sure to check for (absolute or conditional) convergence at each end of the interval, if it is not the set of all real numbers or just $\{0\}$.

ANSWER:

We use the ratio test to look for the radius of convergence. The ratio of successive terms is in absolute value $\frac{x^{n+1}/(n+1)}{x^n/n}$ which simplifies to $x\frac{n}{n+1}$. As $n \to \infty$, the fraction goes to 1, so the limiting ration ρ is x. Hence the series converges absolutely when $|x| < 1$. At the ends of the interval: When $x = 1$, the series is the alternating harmonic series which does converge conditionally. When $x = -1$, the series is the harmonic series except that all the terms are negative, i.e. it is -1 times the harmonic series, and the harmonic series diverges, so this series diverges.

So the interval of convergence is $(-1, 1]$, with conditional convergence at $x = 1$.

(ii) (3 extra credit points) If x is a number for which this series converges absolutely, what is $f(x)$?

ANSWER:

One way to find $f(x)$: If we differentiate this series term-by-term where it converges absolutely, we get $1 - x + x^2 - x^3 - x^4 + \cdots$, a geometric series with first term 1 and ratio $-x$, so it converges to $\frac{1}{1-(-x)} = \frac{1}{1+x}$. Hence our series must be the integral of $\frac{1}{1+x}$, i.e. $\ln(1+x)$.

Another way: The table of "Frequently used Taylor series" on page 831 in the text gives this as $\ln(1+x)$.

(b) Using the formula for coefficients that involves derivatives, find the Maclaurin series for $f(x) = \sin x + \cos x$. (This is not the same f as in part (a).) Show your work,

Either write a formula for the n^{th} term in general or write out all terms through the one involving x^8.

ANSWER:

The derivatives of $\sin x + \cos x$ are successively $\cos x - \sin x$, $-\sin x - \cos x$, $-\cos x + \sin x$, and back to $\sin x + \cos x$. At $x = 0$ those have the values 1, 1, -1, and -1. So the coefficients in the series start out as $\frac{1}{0!}, \frac{1}{1!}, \frac{-1}{2!}, \frac{-1}{3!}, \frac{1}{4!}, \frac{1}{5!}, \frac{-1}{6!}, \frac{-1}{7!}, \frac{1}{8!}$. The exam said that listing the terms through x^8 would suffice, so an acceptable answer is $1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 - \frac{1}{6!}x^6 - \frac{1}{7!}x^7 + \frac{1}{8!}x^8$.

If you want to write the general n^{th} term, you can either construct a formula that differs according to whether n is odd or n is even, or write two parts and add them. Something like this is needed to get that $+, +, -, -,$ pattern: If we let $a_n = \frac{(-1)^{\frac{n}{2}}}{n!}$ when n is even, and $a_n = \frac{(-1)^{\frac{n-1}{2}}}{n!}$ when n is odd, then the series is $\sum_{n=0}^{\infty} a_n x^n$.

Problem 5

(a) For $f(x) = 3 + x - 2x^2 + 4x^3$: Find the Taylor series for $f(x)$ at $a = 1$.

Write out all the non-zero terms or, if there are infinitely many, describe a pattern that they follow.

ANSWER:

Since this is a polynomial of degree 3, its derivatives beyond the fourth will be zero. So there are not infinitely many non-zero terms, and in fact we could write the Maclaurin series as just the polynomial itself. But we are asked for the Taylor series at $a = 1$, not at $a = 0$. You could use that Maclaurin series and by some algebra rewrite it in powers of $x - 1$, but it is probably easier just to compute the coefficients for the Taylor series directly. The first several derivatives are $3 + x - 2x^2 + 4x^3$, $1 - 4x + 12x^2$, $-4 + 24x$, 24, and the rest are all zero. Evaluating those at $x = 1$ we get 6, 9, 20, and 24. Hence the first four coefficients for the Taylor series are $\frac{6}{0!} = 6$, $\frac{9}{1!} = 9$, $\frac{20}{2!} = 10$, and $\frac{24}{3!} = 4$, and all other coefficients are 0. So the series is $6 + 9(x-1) + 10(x-1)^2 + 4(x-1)^3$. (As a check, if you expand those powers and collect terms you get back the original polynomial.)

(b) We know that the area in a circle of radius r is given by the formula $A = \pi r^2$.

Set up and evaluate an integral in polar coordinates to derive that formula.

ANSWER:

We need to integrate $\frac{1}{2}r^2\,d\theta$ through the appropriate region. This circle has constant radius r, and letting θ go from 0 to 2π takes us once around the circle. So we can use $\displaystyle\int_0^{2\pi} \frac{1}{2}r^2\,d\theta$, where r is a constant. That gives $\frac{1}{2}r^2 \displaystyle\int_0^{2\pi} d\theta = \frac{1}{2}r^2(2\pi - 0) = \pi\, r^2$.

Problem 6

Find the general solution of

$$y'' + 2y' - 3y = -30\sin 3x.$$

ANSWER:

First we find y_h, the general solution to the homogeneous equation $y'' + 2y' - 3y = 0$. The characteristic equation is $r^2 + 2r - 3 = 0$ or $(r+3)(r-1) = 0$, so the roots are $r = -3$ and $r = 1$. Hence we can write $y_h = c_1 e^{-3x} + c_2 e^x$.

Now we need to find a solution y_p of the real equation. We use the function on the right hand side, $-30\sin 3x$, and the method of Undetermined Coefficients. $3i$ is not a root of the characteristic equation, those roots were -3 and 1. So we try $y = A\cos 3x + B\sin 3x$ where the coefficients A and B are yet to be determined. Taking derivatives, $y' = -3A\sin 3x + 3B\cos 3x$ and $y'' = -9A\cos 3x - 9B\sin 3x$. Putting those back into the equation, we have $-9A\cos 3x - 9B\sin 3x + 2(-3A\sin 3x + 3B\cos 3x) - 3(A\cos 3x + B\sin 3x) = -30\sin 3x$. If we collect together all of the $\cos 3x$ terms on the left and also all of the $\sin 3x$ terms, we get $(-9A + 6B - 3A)\cos 3x + (-9B - 6A - 3B)\sin 3x = -30\sin 3x$. Simplifying, $(-12A + 6B)\cos 3x + (-6A - 12B)\sin 3x = -30\sin 3x$. Since the $\sin 3x$ terms on both sides must match up, $-6A - 12B = -30$. Likewise the $\cos 3x$ terms, 0 on the right, give us $-12A + 6B = 0$. Solving those two algebraic equations gives $A = 1$ and $B = 2$. So our particular solution $y_p = \cos 3x + 2\sin 3x$.

Putting these together, the general solution is $y = y_p + y_h = \cos 3x + 2\sin 3x + c_1 e^{-3x} + c_2 e^x$. (Believe it or not, if you put that back into the original equation it works!)

Problem 7

Suppose we use $x - \frac{x^3}{6} + \frac{x^5}{120}$ (the first few terms of the Maclaurin series for $\sin(x)$) as an approximation to $\sin(x)$, for $0 \le x \le \frac{1}{2}$.

(a) What is the maximum possible error, i.e. how much could $\sin(x)$ differ from $x - \frac{x^3}{6} + \frac{x^5}{120}$ on that range of x-values?

ANSWER:

For the x values required, $x \ge 0$ so x, x^3, etc., will all be non-negative. So the series will actually be an alternating series for these x values. Hence we can either use the

Alternating Series Estimation theorem or the remainder term from Taylor's Theorem. I will do it both ways:

As an alternating series, the error is bounded in absolute value by the absolute value of the first omitted term. For this truly to be an alternating series, we must be ignoring the zero terms with even powers of x, so the first omitted term is $-\frac{x^7}{7!}$ and the error is at most the absolute value of that. For any x in $[0, \frac{1}{2}]$, that is at most $\frac{(\frac{1}{2})^7}{7!} = \frac{\frac{1}{128}}{5040} = \frac{1}{128\times 5040} = \frac{1}{645120}$.
Using the Taylor Theorem remainder term, we have to decide what n to use in $R_n(x)$. The polynomial contains the series' terms through the 5^{th} power, so we could use $n = 5$, but it even contains the terms through the 6^{th} power since the 6^{th} power term is 0. So we could use $n = 6$. We will get refined accuracy using $n = 6$. Then $|R_6(x)$ will be

$$\frac{|7^{th} \text{ derivative of } \sin x\text{, evaluated somewhere between 0 and } x| \times |x^7|}{7!}.$$

That derivative will be $\pm \sin$ or $\pm \cos$ but we don't know where it will be evaluated, so the simplest thing is to say it is at most 1. Also, for the x values in question, $|x^7|$ will be at most $(\frac{1}{2})^7$. Putting these together we get exactly the same error bound that we got from the alternating series treatment.

(b) Will the value $x - \frac{x^3}{6} + \frac{x^5}{120}$ be larger or smaller than the actual value of $\sin(x)$ for those x-values?

ANSWER: This time it does make a difference which of the above methods we use: We have not had any way of telling from Taylor's theorem which direction the error goes, only how big it is. But we can use the Alternating Series Estimation Theorem. That first omitted term, $-\frac{x^7}{7!}$, is negative for all of the x values involved, so it would have made the estimate smaller to include one more term. Hence the approximation is larger than the actual value.

Problem 8

For each series, tell whether it converges or diverges. Be sure to give a reason based on one of the convergence tests we have studied. If you make algebraic changes in the terms be sure to show what you have done, e.g., to justify a comparison.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{n^3+2}$

ANSWER:

I think the easiest way to do this one is comparison: If we take the term $\frac{3n}{n^3+2}$ and divide numerator and denominator by n we get $\frac{3}{n^2+\frac{2}{n}}$. We can factor the 3 out of the whole series and have $3\sum \frac{1}{n^2+\frac{2}{n}}$. The denominator is bigger than n^2, so the term is less than $\frac{1}{n^2}$, and we know $\sum \frac{1}{n^2}$ converges as a p-series with $p = 2 > 1$. Hence this series converges.

(b) $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n}$

ANSWER:

$\cos \pi n$ is either ± 1, depending on whether n is odd or even, so this is either the alternating harmonic series or its negative: It is actually the negative of the alternating harmonic series, but either way we know it converges.

(c) $\sum_{n=1}^{\infty} \frac{2n}{n^2+5}$

ANSWER:

If we divide numerator and denominator by n, as in part (a), this "looks a lot like" $\frac{1}{n}$, but unfortunately the $+5$ in the denominator makes the fraction smaller so it is not easy to use a comparison. But the numerator is exactly the derivative of the denominator, if we replace n by a continuously varying x, so this is all set up to use the integral test. The improper integral

$$\int_1^{\infty} \frac{2x}{x^2+5} dx = \lim_{b\to\infty} \int_1^b \frac{2x}{x^2+5} = \lim_{b\to\infty} \Big[|\ln(x^2+5)| \Big]_1^b = \lim_{b\to\infty} [\ln(b^2+5) - \ln 6]$$

diverges, so the series diverges.