Mathematics 222, Spring 2008 Lecture 1 (Wilson)

First Midterm Exam February 28, 2008 ANSWERS

NOTE: Many of the integral answers that include trigonometric functions could be written in other forms, using various trig identities. I have not tried to show all of the correct versions!

Problem 1 (18 points)
Evaluate the integrals:

(a) $\int e^{2x} \sin(x)\, dx$.

ANSWER:
This looks like a candidate for integration by parts. You could either let $u = e^{2x}$ and $dv = \sin(x)dx$ or let $u = \sin(x)$ and $dv = e^{2x}dx$. I'll make the first choice. If $u = e^{2x}$, then $du = 2e^{2x}dx$. With $dv = \sin(x)dx$, $v = -\cos(x)$. So the original integral can be rewritten as $-e^{2x}\cos(x) + 2\int e^{2x}\cos(x)dx$. This latter integral needs parts again. I will choose $u = e^{2x}$ again, with $dv = \cos(x)dx$, so $du = 2e^{2x}dx$ and $v = \sin(x)$. Using these on the above partial result, we get $\int e^{2x}\sin(x)dx = -e^{2x}\cos(x) + 2e^{2x}\sin(x) - 4\int e^{2x}\sin(x)dx$.

Now add $4\int e^{2x}\sin(x)dx$ to both sides and get $5\int e^{2x}\sin(x)dx = -e^{2x}\cos(x) + 2e^{2x}\sin(x)$, divide by 5, and remember this is an indefinite integral so we need the constant, to get $\int e^{2x}\sin(x)dx = -\frac{1}{5}e^{2x}\cos(x) + \frac{2}{5}e^{2x}\sin(x) + C$.

(b) $\int \sec^4(x)dx$.

ANSWER:
I will separate $\sec^4(x)$ as $\sec^2(x)\ \sec^2(x)$ and use the identity $\sec^2(x) = 1+\tan^2(x)$ on one of the factors. We have $\int \sec^4(x)dx = \int (1+\tan^x(x))\sec^2(x)dx = \int \sec^2(x)dx + \int \tan^2(x)\sec^2(x)dx$. If we remember that the derivative of the tangent is the square of the secant, the first is trivial. The second is not far behind: We can let $u = \tan(x)$ and have $du = \sec^2(x)dx$. So we get $\tan(x) + \frac{1}{3}\tan^3(x) + C$.

Problem 2 (18 points)

(a) Evaluate $\int_0^2 \frac{2x\, dx}{\sqrt{4-x^2}}$.

ANSWER:
The first thing we must notice is that this is an improper integral. At $x = 2$ the denominator of the integrand goes to zero. So we must use a limit, e.g. $\int_0^2 \frac{2x\, dx}{\sqrt{4-x^2}} = \lim_{b\to 2^-}\int_0^b \frac{2x\, dx}{\sqrt{4-x^2}}$.

Now $2x$ is almost the derivative of $4-x^2$, so we can let $u = 4-x^2$ and have $\int \frac{2x\,dx}{\sqrt{4-x^2}} = -\int u^{-\frac{1}{2}}du = -2u^{\frac{1}{2}} + C$. So we must evaluate $\lim_{b\to 2^-}\left[-2\sqrt{4-x^2}\right]_0^b$. At $x=0$ the quantity inside the brackets gives $-2\sqrt{4} = -4$. At $x=b$ it gives $2\sqrt{4-b^2}$ and as $b \to 2^-$ that has limit 0. So the answer is $0-(-4) = 4$.

(b) Five measured values of a function $f(x)$ are given in the table at the right. Use the trapezoidal rule to approximate $\int_3^5 f(x)\,dx$. You do not need to calculate an error estimate.

x	$f(x)$
3	2
3.5	1
4	3
4.5	4
5	6

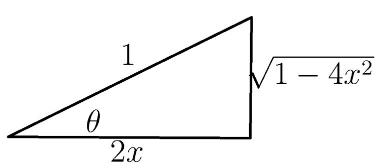
ANSWER:
We can do this straight from the formula for the trapezoidal rule. The interval $[3,5]$ is divided into four intervals, each of length $\Delta x = \frac{1}{2}$. The values of $f(x)$ at the division points are the numbers in the second column of the table. So we have
$\int_3^5 f(x)dx \approx \frac{\Delta x}{2}\left(f(3) + 2f(3.5) + 2f(4) + 2f(4.5) + f(5)\right) = \frac{1}{4}(2+2+6+8+6) = \frac{24}{4} = 6.$

Problem 3 (18 points)

(a) Evaluate $\int_0^{\frac{1}{2\sqrt{2}}} \frac{2\,dx}{\sqrt{1-4x^2}}$.

ANSWER:
This might look like an improper integral, but the quantity inside the square root sign will be positive so ong as $4x^2 < 1$, $x^2 < \frac{1}{4}$, or for positive values of x, $x < \frac{1}{2}$. Since $\frac{1}{2\sqrt{2}} < \frac{1}{2}$ we are safe!

I will use a trigonometric substitution to evaluate this integral, starting by drawing a right triangle: From the triangle we infer $2x = \cos\theta$, so $x = \frac{1}{2}\cos\theta$ and $dx = -\frac{1}{2}\sin\theta\,d\theta$, and



$\sqrt{1-4x^2} = \sin\theta$. Hence we can rewrite the integral as $-\int \frac{\sin\theta\,d\theta}{\sin\theta} = -\int d\theta$, with the new limits of integration yet to be found. Looking at the triangle again, if $x \to 0$, the triangle tends toward being just two vertical lines, so $\theta \to \frac{\pi}{2}$. At $x = \frac{1}{2\sqrt{2}}$, the two legs of the triangle each become $\frac{\sqrt{2}}{2}$, so the triangle becomes an isoceles right triangle and $\theta = \frac{\pi}{4}$. So we now have $-\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} d\theta = -\left(\frac{\pi}{4} - \frac{\pi}{2}\right) = \frac{\pi}{4}$.

(b) Evaluate $\int x^2 \ln(x)\, dx$.

ANSWER:
We use parts again: If we let $u = \ln(x)$ then $du = \frac{1}{x}dx$ will cancel against part of v. Since $dv = x^2 dx$, $v = \frac{x^3}{3}$ and the integral becomes $\frac{x^3}{3} - \int \frac{x^2}{3}dx = \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C$.

Problem 4 (10 points)
A hyperbola in standard position, its center at the origin and its foci on one of the coordinate axes, passes through the (rectangular coordinates) points $(0, \pm 12)$, and it is asymptotic to the lines $y = \pm\frac{12}{5}x$.

(a) Find an equation for this hyperbola.

ANSWER:
Since its foci are on the y-axis, the equation will have the form $-\frac{x^2}{b^2} + \frac{y^x}{a^2} = 1$. The points where it crosses the y-axis will be $(0, \pm a)$, so $a = 12$. The asymptotes will have slope $\pm\frac{a}{b}$, so $\frac{a}{b} = \frac{12}{5}$: Since we already know $a = 12$, b must be 5. So the equation is $-\frac{x^2}{25} + \frac{y^2}{144} = 1$.

(b) Where are the foci of his hyperbola? (Give specific coordinates.)

ANSWER:
The foci will be at $(0, \pm c)$ where $c^2 = a^2 + b^2 = 144 + 25 = 169$. So $c = \sqrt{169} = 13$ and the foci are $(0, \pm 13)$.

(c) What is the eccentricity of this hyperbola?

ANSWER:
The eccentricity is $\frac{c}{a} = \frac{13}{12}$. Note this is greater than 1, appropriate for a hyperbola.

Problem 5 (10 points)
A curve is described by the equation $\frac{x^2}{100} + \frac{y^2}{36} = 1$.

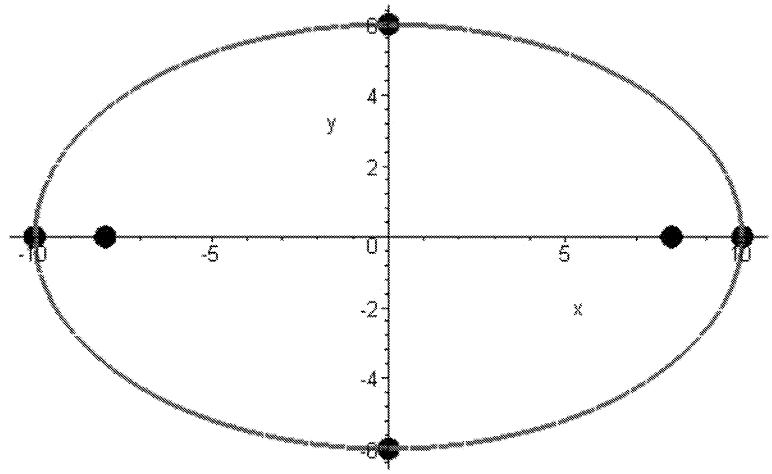
(a) What kind of conic section is this curve?

ANSWER:
This is an ellipse, we can tell from the form of the equation.

(b) Graph the curve: You should label and give specific coordinates for any places where it crosses the $x-$ and $y-$axes, foci, etc. Your graph must not have sharp corners where it should really be smooth, but otherwise you won't be graded on artistic ability.

ANSWER:

Here is a graph as produced by Maple. The foci are shown at $(\pm 8, 0)$, the x-intercept points (formally these would be called the ends of the major axis) are $(\pm 10, 0)$, and the y-intercept points (the ends of the minor axis) are $(0, \pm 6)$.



(c) What is the eccentricity of this curve?

ANSWER:
In the usual notation, $c = 8$ (from the foci) and $a = 10$ (from the ends of the major axis), so the eccentricity $e = \frac{c}{a} = \frac{8}{10} = \frac{4}{5}$.

Problem 6 (12 points)
An object is intended to move so that its position at time t, for $0 \leq t \leq 2\pi$, will be given by $x = 3\cos t$ and $y = 5\sin t$.

(a) Describe this motion using words and/or an equation in rectangular coordinates. Be sure to tell how far up/down and how far left/right the object would move and to tell what kind of conic section it follows. Tell where it starts, what direction it moves, and where it finishes.

ANSWER:
Since $\frac{x}{3} = \cos t$ and $\frac{y}{5} = \sin t$, and $\cos^2 t + \sin^2 t = 1$, we have $\frac{x^2}{9} + \frac{y^2}{25} = 1$ for any value of t. Hence the motion follows along that ellipse. It extends left and right between $(\pm 3, 0)$ and up and down between $(0, \pm 5)$. When $t = 0$, i.e. at the start, the position is $x = 3\cos 0 = 3$ and $y = 5\sin 0 = 0$, so the object starts at the farthest rightmost point $(3, 0)$ where the ellipse crosses the x-axis. As t increases from 0, $\cos t$ decreases so x decreases, and $\sin t$ increases so y increases. Hence the object is moving upward and to the left along the ellipse from that starting point, and it continues on around the ellipse going counter-clockwise. When t reaches 2π the object stops, having made exactly one trip around the ellipse, where it began, at $(3, 0)$.

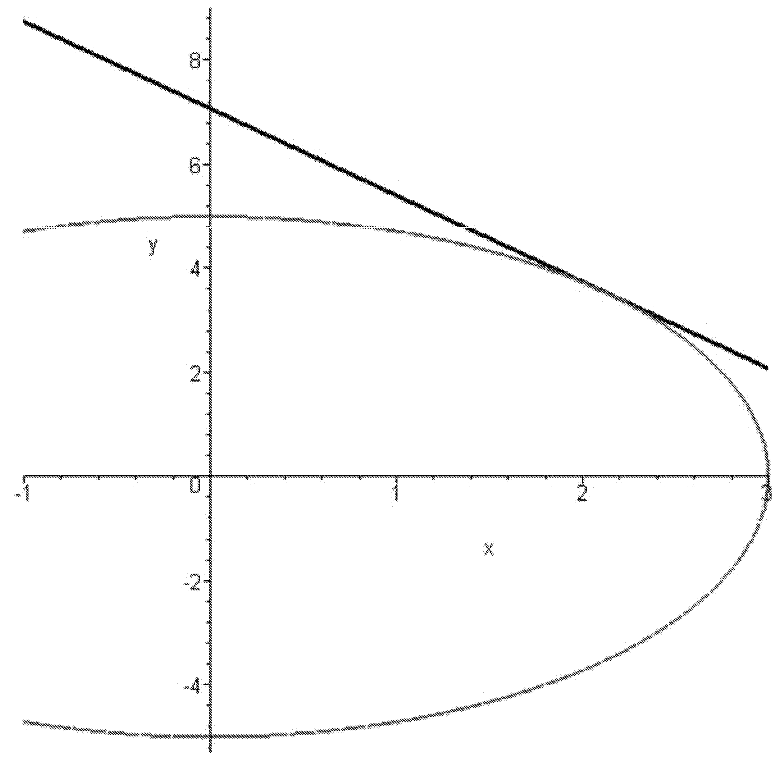
(b) At the instant when $t = \frac{\pi}{4}$, the forces holding the object on that curve fail and the object flies away along the tangent line to the path it had been on.

Find an equation for that tangent line.

ANSWER:
We need to find two things in order to construct the equation for the tangent line: Where was

the object at that time, and what is the slope of the tangent line. "Where" comes from the values of x and y at the time $t = \frac{\pi}{4}$: $x = 3\cos\frac{\pi}{4}$ and $y = 5\sin\frac{\pi}{4}$, so the object is at the point $(\frac{3\sqrt{2}}{2}, \frac{5\sqrt{2}}{2})$. The slope at any value of t is given by $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{5\cos t}{-3\sin t}$. When $t = \frac{\pi}{4}$ both $\cos t$ and $\sin t$ have the value $\frac{\sqrt{2}}{2}$, which will cancel out in the derivative, so $\frac{dy}{dx} = -\frac{5}{3}$.

So now we need an equation for the line through $(\frac{3\sqrt{2}}{2}, \frac{5\sqrt{2}}{2})$ with slope $-\frac{5}{3}$. We can write that as $y - \frac{5\sqrt{2}}{2} = -\frac{5}{3}\left(x - \frac{3\sqrt{2}}{2}\right)$. That can also be simplified as $y = -\frac{5}{3}x + 5\sqrt{2}$. Below is a plot of part of the ellipse with that line superimposed, as a check. Note that I have plotted this with different scales on vertical and horizontal axes in order to reduce the size of the picture!



Problem 7 (14 points)

Consider two curves given in polar coordinates:

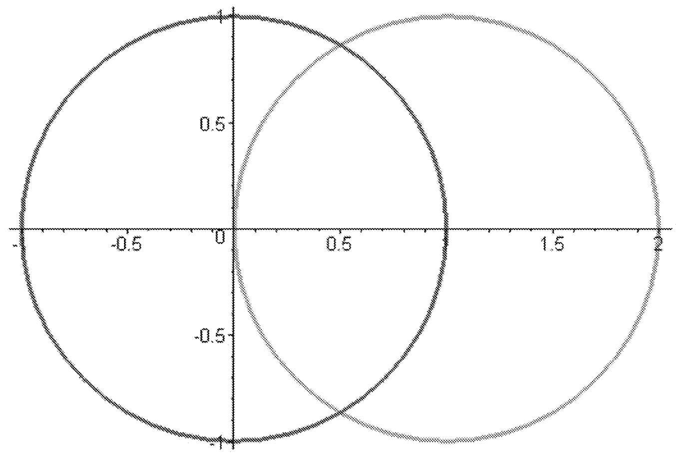
(curve (i)) $r = 1$.

(curve (ii)) $r = 2\cos\theta$.

(a) Find the point(s) where these two curves intersect. Give both the polar and the rectangular coordinates of each intersection point, being sure to label which coordinates are polar and which are rectangular.

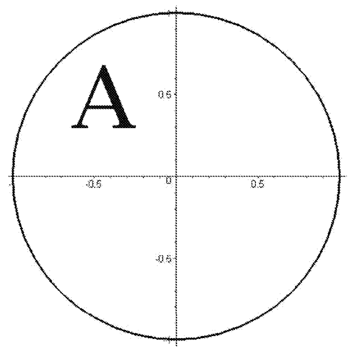
ANSWER:

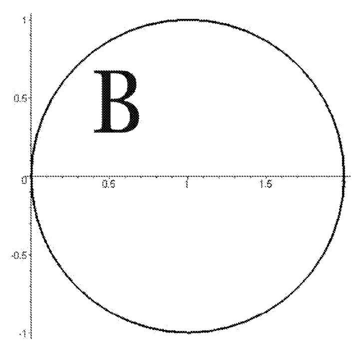
Here is a graph with both curves plotted. There are two intersection points. If we set $r = 1$ and $r = 2\cos\theta$ equal and divide by 2 we have $\cos\theta = \frac{1}{2}$, so $\theta = \pm\frac{\pi}{3}$ or some angle differing from those by a multiple of 2π. At either of those values for θ we have $r = 1$ since the point is on curve (i). You can check that at those values of θ the other curve has $r = 2\cos\theta = 2 \times \frac{1}{2} = 1$, confirming that the points really are on both curves. So in polar coordinates the intersection points are $(1, \frac{\pi}{3})$ and $(1, -\frac{\pi}{3})$.



To convert to rectangular coordinates we use $\sin\pm\frac{\pi}{3} = \pm\frac{\sqrt{3}}{2}$, $x = r\cos\theta$, and $y = r\sin\theta$: $x = 1 \times \frac{1}{2} = \frac{1}{2}$, and $y = 1 \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$, so the coordinates are $(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$.

(b) At the right are two graphs. Which one (A or B) is curve (i), and which one is curve (ii)?





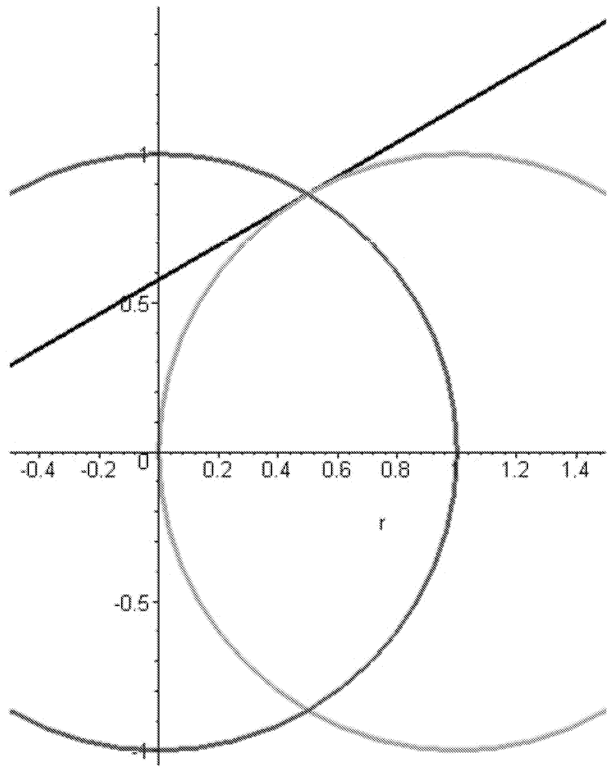
ANSWER:
Curve (i), $r = 1$, is centered at the origin, so that is A, and curve (ii) is B.

(c) One of the intersection points (and only one) is in the first quadrant, where $x \geq 0$ and $y \geq 0$. At that point, find the slope and the tangent line (equation in rectangular coordinates) to curve (ii).

ANSWER:
The intersection point in the first quadrant is, in rectangular coordinates, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. We need to find the slope there, for the tangent to curve (ii) $r = 2\cos\theta$. If we call that $r = f(\theta) = 2\cos\theta$ we can use the formula derived in class for the slope, $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$. At the intersection point: $f(\theta) = r = 1$. $f'(\theta) = -2\sin\theta = -2\sin\frac{\pi}{3} = -2 \times \frac{\sqrt{3}}{2} = -\sqrt{3}$. $\sin\theta = \frac{\sqrt{3}}{2}$. $\cos\theta = \frac{1}{2}$. So the slope is

$$\frac{dy}{dx} = \frac{-\sqrt{3} \times \frac{\sqrt{3}}{2} + 1 \times \frac{1}{2}}{-\sqrt{3} \times \frac{1}{2} - 1 \times \frac{\sqrt{3}}{2}} = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3} \approx 0.5774.$$

The tangent line is $y - \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$ which can be simplified as $y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$. Below is a plot of that line superimposed on the two circles, as a check:



1
0.5
0
-0.5
-1
-0.4
-0.2
0.2
0.4
0.6
0.8
1
1.2
1.4
r