Your Name: ____________________

Circle your TA's name:

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Mathematics 222, Spring 2005   **Lecture 4 (Wilson)**

Final Exam   May 11, 2005

Write your answers to the ten problems in the spaces provided. If you must continue an answer somewhere other than immediately after the problem statement, be sure (a) to tell where to look for the answer, and (b) to label the answer wherever it winds up. In any case, be sure to make clear what is your final answer to each problem.

If you evaluate an improper integral without explicit use of limits, you will receive no credit.

Wherever applicable, leave your answers in exact forms (using $\frac{\pi}{3}$, $\sqrt{3}$, $\cos(0.6)$, and similar numbers) rather than using decimal approximations. If you use a calculator to evaluate your answer be sure to show what you were evaluating!

There are formulas for your use on the last page of the exam.

You may refer to notes you have brought on up to three index cards as announced in class.

BE SURE TO SHOW YOUR WORK, AND EXPLAIN WHAT YOU DID. YOU MAY RECEIVE REDUCED OR ZERO CREDIT FOR UNSUBSTANTIATED ANSWERS. ("I did it on my calculator" and "I used a formula from the book" (without more details) are not sufficient substantiation...)

THERE IS A PROBLEM ON THE BACK OF THIS SHEET!

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
TOTAL	200	

Problem 1 (20 points)

Let $\vec{u} = 2\vec{\imath} + 2\vec{\jmath} - \vec{k}$ and $\vec{v} = -\vec{\imath} + \vec{k}$.

(a) Calculate $\vec{u} \cdot \vec{v}$.

(b) Find the angle θ between $\vec{u}$ and $\vec{v}$.

(c) Find the vector projection of $\vec{u}$ on $\vec{v}$.

(d) Calculate $\vec{u} \times \vec{v}$.

Problem 2 (20 points)
Evaluate the integrals:

(a) $$\int \frac{\sqrt{9-x^2}}{x^2}\,dx$$

(b) $$\int e^{-x}\cos(2x)\,dx$$

Problem 3 (20 points)

(a) For each series: Tell whether it converges absolutely, conditionally, or not at all, and justify your answer.

(i) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\, e^{(1/n)}$

(ii) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n+1)}$

(iii) $\displaystyle\sum_{n=1}^{\infty}\frac{\frac{1}{2}+\cos(2n)}{n^{\frac{3}{2}}}$

(b) Find the radius and interval of convergence for the power series $\displaystyle\sum_{n=1}^{\infty}\frac{(2x)^n}{n^2\, 3^n}$.

Problem 4 (20 points)
Label three points in space as $P = (2, 1, 3)$, $Q = (2, 2, 5)$, and $R = (1, 1, 6)$.

(a) Find an equation for the plane passing through points P, Q, and R.

(b) Find equations in both parametric and symmetric forms for the line that goes through Q and is perpendicular to the plane you found in (a).

Problem 5 (20 points)
Consider the two circles $r = 3\sin\theta$ and $r = -\sqrt{3}\sin\theta$: Find the area of the region which is inside both of these circles. If you make use of any symmetries be sure to say exactly what you are doing, but be careful because some things that look symmetric may not really be so.

Problem 6 (20 points)

(a) Find the solution of the differential equation $y'' - 24y + 169 = 0$ that satisfies $y(0) = -1$ and $y'(0) = 17$.

(You may find useful the numeric fact that $(24)^2 - 4 \times 169 = -100$.)

(b) For the differential equation $y'' - y' - 6y = 0$, the roots of the auxiliary equation are $r_1 = -2$ and $r_2 = 3$. Find all solutions of the non-homogeneous equation

$$y'' - y' - 6y = -10e^{3x}.$$

Problem 7 (20 points)
Evaluate:

$$\int_{-2}^{2} \frac{dt}{\sqrt{4-t^2}}$$

Problem 8 (20 points)

The position vector of a particle is given at time t by $\vec{r}(t) = (2t - t^2)\vec{\imath} + t\vec{\jmath} + e^t\vec{k}$.

(a) Find the velocity vector $\vec{v}(t)$ as a function of t.

(b) Find the acceleration vector $\vec{a}(t)$ as a function of t.

(c) Find equations for the tangent line to the path of this particle at the instant when $t = 0$. (You may use either parametric or symmetric form for the line.)

Problem 9 (20 points)
A conic section has the following properties:

1. It is symmetric about the y-axis and the x-axis.
2. It crosses the y-axis at $y = \pm 3$.
3. It has foci at $(0, \pm 5)$.

(a) What kind of curve is it? (Circle, Ellipse, Parabola, Hyperbola?) How do you know?

(b) Find an equation for this conic section. Your equation should contain only x's and y's and numbers, i.e. you should find values for all parameters.

(c) Tell what other facts you can about this curve: If it is a circle be sure to include center and radius. If it is an ellipse be sure to include how long the axes are and which is horizontal, and where it crosses the coordinate axes. If it is a parabola be sure to describe its orientation, e.g. opening to the left. If it is a hyperbola be sure to include equations for the asymptotes and tell whether it opens vertically or horizontally. In any case give the eccentricity.
You are welcome to use a labelled sketch as a way of organizing your answer, but you will not be graded on your artistic ability so long as the necessary facts are shown.

Problem 10 (20 points)
The terms of the Maclaurin series for $\sin(x)$ through $\frac{x^8}{8!}$ are going to be used to approximate $\sin(x)$ for $x \in (-\frac{1}{2}, \frac{1}{2})$.
How big might the error be in this approximation? Be sure to justify your answer using theorems we have had in this course!

Some formulas, identities, and numeric values you might find useful:

Values of trig functions:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	—

Derivative formulas:

1. $\frac{d}{dx}\tan x = \sec^2 x$
2. $\frac{d}{dx}\sec x = \sec x\,\tan x$
3. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
4. $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
5. $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$
6. $\frac{d}{dx}\ln x = \frac{1}{x}$

Trig facts:

1. $\sec^2\theta = \tan^2\theta + 1$
2. $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
3. $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
4. $\sin^2 x = \frac{1}{2}(1-\cos 2x)$
5. $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

Integral formulas:

1. $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$
2. $\int \frac{du}{1+u^2} = \tan^{-1} u + C$
3. $\int \sec(u)\,du = \ln|\sec(u) + \tan(u)| + C$
4. $\int u\,dv = uv - \int v\,du$

Trial functions for Undetermined Coefficients:

For a term in $f(x)$ which is a multiple of	If	Then use a term like
$\sin(kx)$ or $\cos(kx)$	ki is not a root of the auxiliary equation ki is a root of the auxiliary equation	$A\cos(kx) + B\sin(kx)$ $Ax\cos(kx) + Bx\sin(kx)$
e^{nx}	n is not a root of the auxiliary equation n is a single root of the auxiliary equation n is a double root of the auxiliary equation	Ce^{nx} $Cx\,e^{nx}$ $Cx^2\,e^{nx}$
A polynomial $ax^2 + bx+c$ of degree at most 2	0 is not a root of the auxiliary equation 0 is a single root of the auxiliary equation 0 is a double root of the auxiliary equation	a polynomial $Dx^2 + Ex + F$ of the same degree as $ax^2 + bx + c$ a polynomial Dx^3+Ex^2+Fx of degree one more a polynomial $Dx^4 + Ex^3 + Fx^2$ of degree two more