Mathematics 222 **Lecture 4** **Wilson**

Final Exam **May 11, 2005** **Answers**

Problem 1
Let $\vec{u} = 2\vec{\imath} + 2\vec{\jmath} - \vec{k}$ and $\vec{v} = -\vec{\imath} + \vec{k}$.

(a) Calculate $\vec{u} \cdot \vec{v}$.

ANSWER: $\vec{u} \cdot \vec{v} = 2 \times (-1) + 2 \times 0 + (-1) \times 1 = -3$.

(b) Find the angle θ between $\vec{u}$ and $\vec{v}$.

ANSWER: $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$. $\vec{u} = \sqrt{4+4+1} = 3$ and $\vec{v} = \sqrt{1+1} = \sqrt{2}$, so $\cos\theta = \frac{-3}{3\sqrt{2}} = -\frac{\sqrt{2}}{2}$ and $\theta = \frac{3\pi}{4}$.

(c) Find the vector projection of $\vec{u}$ on $\vec{v}$.

ANSWER: The vector projection of $\vec{u}$ on $\vec{v}$ can be calculated in several ways. Thinking it through, instead of just using a formula, we are looking for a vector lying along $\vec{v}$ whose magnitude is the scalar projection of $\vec{u}$ onto $\vec{v}$. The scalar projection of $\vec{u}$ onto $\vec{v}$ is $\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$ which is $\frac{-3}{\sqrt{2}}$ or $-\frac{3}{2}\sqrt{2}$. A unit vector in the $\vec{v}$ direction is $\frac{1}{|\vec{v}|}\vec{v}$ which is $\frac{1}{\sqrt{2}}\vec{v}$ or $-\frac{\sqrt{2}}{2}\vec{\imath} + \frac{\sqrt{2}}{2}\vec{k}$. Putting these together we get the vector projection to be $\frac{-3}{\sqrt{2}} \times (-\frac{\sqrt{2}}{2}\vec{\imath} + \frac{\sqrt{2}}{\vec{k}})$ which simplifies to $\frac{3}{2}\vec{\imath} - \frac{3}{2}\vec{k}$.

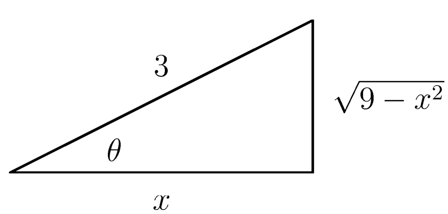
(d) Calculate $\vec{u} \times \vec{v}$.

ANSWER: Set up and evaluate the determinant

$$\vec{u} \times \vec{v} = det \begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix} = 2\vec{\imath} - \vec{\jmath} + 2\vec{k}.$$

Problem 2
Evaluate the integrals:

(a) $$\int \frac{\sqrt{9-x^2}}{x^2}\,dx$$



ANSWER: I would draw and label a triangle as above. Now we can set $x = 3\cos\theta$ and $\sqrt{9-x^2} = 3\sin\theta$, so $dx = -3\sin\theta$, and substitute into the integral to get

$$\int \frac{(3\sin\theta)(-3\sin\theta\, d\theta)}{9\cos^2\theta} = -\int \frac{\sin^2\theta}{\cos^2\theta} d\theta = -\int \tan^2\theta\, d\theta.$$

Next use $\tan^2\theta = \sec^2\theta - 1$ to get $-\int(\sec^2\theta - 1)\, d\theta = -\tan\theta + \theta + C$. Now we need to convert back to x instead of θ: We can read $\tan\theta$ from the triangle, and get $-\frac{\sqrt{9-x^2}}{x} + \arcsin\frac{\sqrt{9-x^2}}{3} + C$. (There are other, equivalent, ways of writing the answer.)

(b) $\int e^{-x} \cos(2x)\, dx$

ANSWER: This begs for integration by parts. If we let $u = \cos(2x)$ and $dv = e^{-x}dx$, $du = -2\sin(2x)dx$ and $v = -e^{-x}$. Using the integration by parts formula we have $uv - \int v\, du = -e^{-x}\cos(2x) - 2\int e^{-x}\sin(2x)\, dx$. We apply integration by parts again to this latter integral, with $u = \sin(2x)$ and $dv = e^{-x}dx$ so $du = 2\cos(2x)dx$ and $v = -e^{-x}$. Now we have $-e^{-x}\cos(2x) - 2(uv - \int v\, du) = -e^{-x}\cos(2x) - 2(-e^{-x}\sin(2x) + 2\int e^{-x}\cos(2x)dx)$. Expanding this and recalling the original integral we have $\int e^{-x}\cos(2x)\, dx = -e^{-x}\cos(2x) + 2e^{-x}\sin(2x) - 4\int e^{-x}\cos(2x)\, dx$. Solving for the integral, and noticing that we need "$+C$" that got omitted in going from dv to v, we get $\int e^{-x}\cos(2x)\, dx = \frac{1}{5}e^{-x}\left(-\cos(2x) + 2\sin(2x)\right) + C$.

Problem 3

(a) For each series: Tell whether it converges absolutely, conditionally, or not at all, and justify your answer.

(i) $\sum_{n=1}^{\infty} (-1)^n e^{(1/n)}$

ANSWER: This is an alternating series and the terms are decreasing, but they are decreasing with limit $e^0 = 1$ and not 0. Hence this series does not converge.

(ii) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

ANSWER: The denominators are getting arbitrarily large so the fractions decrease with limit zero, so this alternating series converges. But if we make the signs positive we have a series whose terms are larger than $\frac{1}{n}$ and so diverges. Hence this series converges conditionally.

(iii) $\sum_{n=1}^{\infty} \frac{\frac{1}{2} + \cos(2n)}{n^{\frac{3}{2}}}$

ANSWER: If we take the absolute values of these terms, the numerator is never greater than $1\frac{1}{2}$ and the denominator is $n^{\frac{3}{2}}$. That series converges by comparison to the p-series with terms $\frac{1}{n^{\frac{3}{2}}}$, which converges since $p = \frac{3}{2} > 1$. Hence this series converges absolutely.

(b) Find the radius and interval of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(2x)^n}{n^2\, 3^n}$.

ANSWER: Taking the absolute value of the ratio of succesive terms $\frac{a_{n+1}}{a_n}$, and simplifying algebraically, we get $\frac{2}{3}|x|\frac{n+1}{n}$. The limit of this ratio as $n \to \infty$ is $\frac{2}{3}|x|$ so the series converges absolutely when $\frac{2}{3}|x| < 1$, i.e. $-\frac{3}{2} < x < \frac{3}{2}$. Hence the radius of convergence is $\frac{3}{2}$. Now we need to check whether the series converges or diverges at the ends of the interval. At $x = -\frac{3}{2}$, the series amounts to $\sum(-1)^n \frac{1}{n^2}$, and at $x = \frac{3}{2}$ the series amounts to $\sum \frac{1}{n^2}$. Since the second of those is a p-series with $p = 2 > 1$ it converges, and the second series is just the absolute values of the first, so both converge. Hence the interval of convergence is $[-\frac{3}{2}, \frac{3}{2}]$.

Problem 4

Label three points in space as $P = (2, 1, 3)$, $Q = (2, 2, 5)$, and $R = (1, 1, 6)$.

(a) Find an equation for the plane passing through points P, Q, and R.

ANSWER: First we find a vector $\vec{n}$ perpendicular to the plane. We do this by finding two vectors in the plane and constructing their cross product. For vectors in the plane I will use the vector from P to Q and the vector from P to R, but other pairs would work also. If we let $\vec{u}$ be the vector from P to Q, $\vec{u} = 0\vec{\imath} + \vec{\jmath} + 2\vec{k}$. Similarly the vector from P to R is $\vec{v} = -\vec{\imath} + 0\vec{\jmath} + 3\vec{k}$. Now $\vec{n} = \vec{u} \times \vec{v} = 3\vec{\imath} - 2\vec{\jmath} + \vec{k}$.

Next we find the plane perpendicular to $\vec{n}$ passing through any one of the points P, Q, or R: I will use P. The equation for this plane is $3(x-2) - 2(y-1) + 1(z-3) = 0$ which simplifies to $3x - 2y + z = 7$.

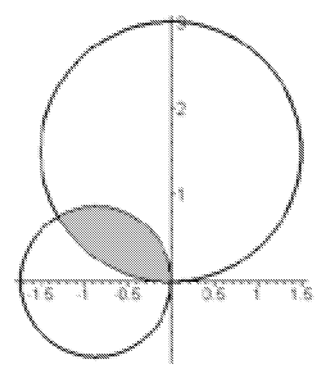
(b) Find equations in both parametric and symmetric forms for the line that goes through Q and is perpendicular to the plane you found in (a).

ANSWER: We already know a vector $\vec{n}$ in the correct direction for the line, so we can immediately write down the parametric equations $x = 3t + 2$, $y = -2t + 2$, and $z = t + 5$. In symmetric form we have $\frac{x-2}{3} = \frac{y-2}{-2} = \frac{z-5}{1}$.

Problem 5

Consider the two circles $r = 3\sin\theta$ and $r = -\sqrt{3}\cos\theta$: Find the area of the region which is inside both of these circles.

ANSWER: The shaded region at the right has the area we need to calculate. It may look symmetric about the tilted line connecting its "corners", but note that one circle curves more sharply than the other: we really need to calculate the whole area as the sum of two integrals.



The two circles intersect where $3\sin\theta = -\sqrt{3}\cos\theta$, $\frac{\sin\theta}{\cos\theta} = \frac{\sqrt{3}}{3}$, $\tan\theta = \frac{\sqrt{3}}{3}$, so $\theta = \frac{5\pi}{6}$. The "upper" part of the shaded region, bounded by the smaller circle, starts with $\theta = \frac{\pi}{2}$ and continues to $\frac{5\pi}{6}$, while the remaining part goes from $\frac{5\pi}{6}$ to π bounded by the larger circle. The area we need can be computed then as

$$\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}}(-\sqrt{3}\cos\theta)^2\,d\theta+\frac{1}{2}\int_{\frac{5\pi}{6}}^{\pi}(3\sin\theta)^2\,d\theta=\frac{3}{2}\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}}\cos^2\theta\,d\theta+\frac{9}{2}\int_{\frac{5\pi}{6}}^{\pi}\sin^2\theta\,d\theta.$$

Using the half-angle formulas for $\sin^2\theta$ and $\cos^2\theta$ this can be evaluated as $\frac{5\pi}{8}-\frac{3\sqrt{3}}{4}$.

Problem 6

(a) Find the solution of the differential equation $y''-24y+169=0$ that satisfies $y(0)=-1$ and $y'(0)=17$.

ANSWER: The auxiliary equation $r^2-24r+169=0$ has roots $r=\frac{24\pm\sqrt{-100}}{2}=12\pm5i$. Hence the general solution is $y(x)=e^{12x}(C_1\cos 5x+C_2\sin 5x)$. Taking the derivative, $y'(x)=e^{12x}(12C_1\cos 5x+12C_2\sin 5x-5C_1\sin 5x+5C_2\cos 5x)$.

Putting in the initial conditions we have

$C_1=-1$ and then, using that fact, $17=-12+5C_2$ so $C_2=\frac{29}{5}$. Thus the desired solution is $y(x)=e^{12x}(-\cos 5x+\frac{29}{5}\sin 5x)$.

(b) For the differential equation $y''-y'-6y=0$, the roots of the auxiliary equation are $r_1=-2$ and $r_2=3$. Find all solutions of the non-homogeneous equation

$$y''-y'-6y=-10e^{3x}.$$

ANSWER: From the roots of the auxiliary equation we know the homogeneous solution is $y_h=C_1e^{-2x}+C_2e^{3x}$. We need to find a particular solution y_p to the nonhomogeneous solution. Since 3 is a single root of the auxiliary equation we try $y_p=Cxe^{3x}$ for some constant C. Then $y_p'=Ce^{3x}+3Cxe^{3x}$ and $y_p''=6Ce^{3x}+9Cxe^{3x}$. Putting those back into $y''-y'-6y$ we get $5Ce^{3x}$, so C must be -2 in order to give $-10e^{3x}$. Hence $y_p=-2xe^{3x}$ and the complete solution is $y_p+y_h=-2xe^{3x}+C_1e^{-2x}+C_2e^{3x}$.

Problem 7
Evaluate:

$$\int_{-2}^{2}\frac{dt}{\sqrt{4-t^2}}$$

ANSWER: This is an improper integral: The integrand "blows up" with denominator zero at each end of the interval of integration. We rewrite the integral as $\lim_{a\to-2^+}\int_a^0\frac{dt}{\sqrt{4-t^2}}+\lim_{b\to2^-}\int_0^b\frac{dt}{\sqrt{4-t^2}}$. (You could break at some point other than 0.) The integral $\int\frac{dt}{\sqrt{4-t^2}}$ gives $\arcsin\frac{t}{2}$, either from a formula or by drawing a triangle. Thus we have $\lim_{a\to-2^+}(\arcsin(0)-\arcsin(\frac{a}{2}))+\lim_{b\to2^-}(\arcsin(\frac{b}{2})-\arcsin(0))=(0-(-\frac{\pi}{2}))+(\frac{\pi}{2}-0)=\pi$.

Problem 8
The position vector of a particle is given at time t by $\vec{r}(t) = (2t - t^2)\vec{\imath} + t\vec{\jmath} + e^t\vec{k}$.

(a) Find the velocity vector $\vec{v}(t)$ as a function of t.

ANSWER: $\vec{v}(t) = \vec{r}\,'(t) = (2 - 2t)\vec{\imath} + \vec{\jmath} + e^t\vec{k}$.

(b) Find the acceleration vector $\vec{a}(t)$ as a function of t.

ANSWER: $\vec{a}(t) = \vec{v}\,'(t) = -2\vec{\imath} + e^t\vec{k}$.

(c) Find equations for the tangent line to the path of this particle at the instant when $t = 0$.

ANSWER: When $t = 0$, $\vec{v}(0)$ gives $2\vec{\imath} + \vec{\jmath} + \vec{k}$ as a vector in the direction of the tangent line. (Note: When I first posted these I had an error here, I had $2\vec{\jmath}$ rather than $\vec{\jmath}$...) At that instant the position is given by $\vec{r}(0) = \vec{k}$, i.e. the particle is at the point $(0, 0, 1)$. In parametric form the tangent line is $x = 2t + 0 = 2t$, $y = t + 0 = 2t$, and $z = t + 1$. In symmetric form we have $\frac{x}{2} = \frac{y}{1} = \frac{z-1}{1}$.

Problem 9
A conic section has the following properties:

1. It is symmetric about the y-axis and the x-axis.
2. It crosses the y-axis at $y = \pm 3$.
3. It has foci at $(0, \pm 5)$.

(a) What kind of curve is it? (Circle, Ellipse, Parabola, Hyperbola?) How do you know?

ANSWER: This must be a hyperbola since its foci are further from the center $((0, \pm 5))$ than the points where it crosses the axis they line on $((0, \pm 3))$.

(b) Find an equation for this conic section. Your equation should contain only x's and y's and numbers, i.e. you should find values for all parameters.

ANSWER: The equation will be of the form $-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, since this is a hyperbola opening up and down. Putting in the fact that $y = \pm 3$ when $x = 0$ we get $a = 3$. We know that the parameters a, b, and c satisfy $a^2 + b^2 = c^2$ where c is the distance to the foci, i.e. $c = 5$. Hence we have $9 + b^2 = 25$ so $b = 4$. Thus the equation is $-\frac{x^2}{16} + \frac{y^2}{9} = 1$.

(c) Tell what other facts you can about this curve: If it is a circle be sure to include center and radius. If it is an ellipse be sure to include how long the axes are and which is horizontal, and where it crosses the coordinate axes. If it is a parabola be sure to describe its orientation, e.g. opening to the left. If it is a hyperbola be sure to include equations for the asymptotes and tell whether it opens vertically or horizontally. In any case give the eccentricity.

As a hyperbola opening up and down the main things we can give about it are its eccentricity and asymptotes. The eccentricity is $\frac{c}{a} = \frac{5}{3}$. The asymptotes are the lines $y = \pm\frac{3}{4}x$.

Problem 10

The terms of the Maclaurin series for $\sin(x)$ through $\frac{x^8}{8!}$ are going to be used to approximate $\sin(x)$ for $x \in (-\frac{1}{2}, \frac{1}{2})$.

How big might the error be in this approximation? Be sure to justify your answer using theorems we have had in this course!

ANSWER: You can answer this using either the remainder term from Taylor's theorem or the alternating series error estimation process. Since for any x the odd powers of x that appear in the series will either all be positive or all negative, the alternating signs in the Maclaurin series for $\sin(x)$ give an alternating series. The terms we are to use actually go through $\frac{x^7}{7}$ since the term in the $\sin(x)$ Maclaurin series with x^8 has coefficient zero: We can take advantage of this to sneak some extra accuracy.

I will use the Taylor's theorem remainder term. That has the form $R_n(x) = \frac{f^{(n+1)}(c)\,x^{n+1}}{(n+1)!}$ where $f^{(n+1)}(c)$ means the $(n+1)^{st}$ derivative of $f(x) = \sin(x)$ evaluated at some number c between 0 and x. In the case we care about, n seems to be 7: We can actually do better since the 8^{th} degree term is 0 and so the polynomial we are to use is actually the same as the one with $n = 8$. I will use $n = 8$ and hence $n + 1 = 9$. Since all derivatives of $\sin(x)$ are $\pm\sin(x)$ or $\pm\cos(x)$, the largest that $f^9(c)$ can be is 1. For $x \in (-\frac{1}{2}, \frac{1}{2})$, the largest that x^9 can be in absolute value is $\left(\frac{1}{2}\right)^9 = \frac{1}{512}$. Hence the size (absolute value) of $R_8(x)$ is at most $\frac{1}{512\times(9!)}$. That is a satisfactory answer, or you can approximate it as 5.3823×10^{-9}.