Mathematics 234, Fall 2004 Lecture 3 (Wilson)

Second Midterm Exam April 12 2005 ANSWERS

Problem 1

The series $\sum_{n=0}^{\infty}(-1)^n \frac{2n}{3^n+1}$ does converge, to some number L.

(a) Give a reason, based on some theorem we have studied, that assures us this series really does converge. Be sure to show why the theorem applies.

ANSWER: This is an alternating series, and the magnitude of the terms consistently decreases with limit 0. Hence the series converges by the alternating series test.

(b) If we add up only the terms of this series corresponding to $n = 0$ through $n = 9$, how far can the resulting sum differ from L?

ANSWER: For a convergent alternating series, the "error" due to truncation is the size of the first term omitted. In this case that is the term with $n = 10$, $\frac{20}{3^{10}+1}$. That is approximately 0.0003387, but the first form is more accurate.

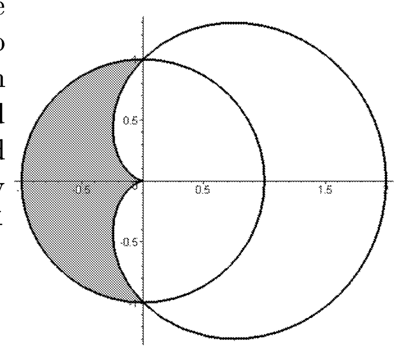
(c) If we use the sum from (b), just the first ten terms, will that sum be larger than L or will it be smaller than L? Be sure to give a reason for your answer!

ANSWER: Since this is a converging alternating series, the sum of the terms through $n = 9$ will be larger than L if the first omitted term is negative, or smaller if the first omitted term is positive. The sign on the $n = 10$ term is given by $(-1)^{10}$ which is positive, so the sum of the ten terms will be less than L.

Problem 2

Find the area of the region which is inside the circle $r = 1$ but outside the cardioid $r = 1 + \cos\theta$. Be sure to show how you find the limits of integration: Explain what (if any) symmetries you are using, show how you find particular θ values that matter, etc. A sketch may be helpful, but you should not just read numbers from a picture.

ANSWER: The region is shaded in the picture to the right. We can guess from the picture that the curves intersect on the vertical axis, both above and below the origin. Solving to find where they meet, $1 = 1 + \cos\theta$ so $\cos\theta = 0$ which occurs for $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$, justifying what we observed in the picture. (The intersections can also be described as $\theta = \pm\frac{\pi}{2}$, but we need to integrate over the range on the left so we will use $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$.) We could either compute the area by an integral for $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ or use symmetry about the horizontal axis. Using the symmetry, we could integrate from $\frac{\pi}{2}$ to π and then double the answer. Integrating over the full range we have



$$\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(1^2 - (1+\cos\theta)^2\right) d\theta.$$

Squaring and simplifying we have $\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (-2\cos\theta - \cos^2\theta)\, d\theta$. Using $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ we can evaluate the integral and get $2 - \frac{\pi}{4}$ which is approximately 1.2146.

Problem 3

An ellipse has one focus at $(0, 4)$ and the other focus at $(12, 4)$, and the length of the major axis (the length from one end of the ellipse to the other end in the longest direction) is 20.

Find an equation for this ellipse.

ANSWER: I will do this in two steps. First I will consider the ellipse with respect to coordinate axes such that the ellipse is in "standard position", i.e. the origin of the axes is at the center of the ellipse and the major axis runs along one of the coordinate axes. Then I will translate the axes to match the description in the problem.

The major axis runs along the $y = 4$ line, since the foci are on that line. The center of the ellipse is 6 units to the right of the y axis, equidistant from the foci. This gives the coordinates of the center, for future reference, as $(6, 4)$. You would not have to be this formal, but we can let u and v be horizontal and vertical coordinates for axes through the center of the ellipse, parallel to the original axes.

In the (u, v) system: The center of the ellipse is at $(0, 0)$; The foci are at $(0, \pm 6)$; The ellipse crosses the u axis at $(0, \pm 10)$, since the major axis is 20 units long. The ellipse in these coordinates must have an equation of the form $\frac{u^2}{(10)^2} + \frac{v^2}{b^2} = 1$, where b is some number not yet determined. But the distance from the center to a focus, which we have frequently called c, is 6, and $c^2 = a^2 - b^2$ where $a = 10$. Thus $b^2 = a^2 - c^2 = 100 - 36 = 64$, so $b = 8$. Thus in the (u, v) system we have the equation as $\frac{u^2}{100} + \frac{v^2}{64} = 1$.

Now we need to translate the coordinates back to the original system which I will call x and y. You can use formulas, $x = u + h$ and $y = v + k$ where h and k are the (x, y) coordinates of the center. Or you can just think it through: We know x will be u plus some constant, such that $u = 0$ where $x = 6$, so $x = u + 6$, and similarly $y = v + 4$. Rewriting those as $u = x - 6$ and $v = y - 4$ and substituting those into the equation we get

$$\frac{(x-6)^2}{100} + \frac{(y-4)^2}{64} = 1.$$

Problem 4

For $f(x) = (x + 1)^{3/2}$, find the first few terms of the Taylor series for $f(x)$ at $a = 8$.

Explicitly write the series through (including) the term of degree 3. You do not have to construct an expression for the n^{th} term in general.

ANSWER: I will organize the calculations in a table, to the right. We need to calculate derivatives through the third derivative, evaluate them at $a = 8$, and divide each by the appropriate factorial, to get the coefficients a_n for $n = 0$ through $n = 3$. Once we have those we can write the series out as

$$27 + \frac{9}{2}(x-8) + \frac{1}{8}(x-8)^2 - \frac{1}{432}(x-8)^3.$$

n	$f^{(n)}$	$f^{(n)}(8)$	a_n
0	$(x+1)^{3/2}$	27	27
1	$\frac{1}{2}(x+1)^{1/2}$	$\frac{9}{2}$	$\frac{9}{2}$
2	$\frac{3}{4}(x+1)^{-1/2}$	$\frac{1}{4}$	$\frac{1}{8}$
3	$-\frac{3}{8}(x+1)^{-3/2}$	$-\frac{1}{72}$	$-\frac{1}{432}$

Problem 5

(a) Find the solution of $y'' - 6y' + 9y = 0$ that satisfies $y(0) = 1$ and $y'(0) = 1$.

ANSWER: First we find all solutions, ignoring the initial conditions. The auxiliary equation is $r^2 - 6r + 9 = 0$, i.e. $(r - 3)^2 = 0$, which has one repeated real root, $r = 3$ and $r = 3$. Hence the general solution is $y(x) = C_1 e^{3x} + C_2 x\, e^{3x}$ for arbitrary constants C_1 and C_2. From that we have $y'(x) = 3C_1 e^{3x} + 3c_2 x\, e^{3x} + C_2 e^{3x}$. Hence $y(0) = C_1$ and $y'(0) = 3C_1 + C_2$. Using $y(0) = 1$ we have $C_1 = 1$. Using $y'(0) = 1$ we have $1 = 3 + C_2$, so $C_2 = -2$. Thus the solution is $y(x) = e^{3x} - 2\, x\, e^{3x}$.

(b) For the <u>different</u> differential equation $y'' + 6y' + 13y = 0$:

Find an expression for <u>all</u> solutions $y(x)$.

ANSWER: This time the auxiliary equation is $r^2 + 6r + 13 = 0$. Using the quadratic formula we have for the roots $r = \frac{-6 \pm \sqrt{36-52}}{2}$ or $r = -3 \pm 2i$. Hence the solutions are $y(x) = e^{-3x}(C_1 \cos 2x + C_2 \sin 2x)$.

<u>Problem 6</u>

For each of the following three series, tell whether it converges Absolutely, Conditionally, or Not At All. Be sure to give reasons for your answers!

(a) $\displaystyle\sum_{n=3}^{\infty} (-1)^n \frac{2n}{3n + \cos(n)}$

ANSWER: The n^{th} term $a_n = (-1)^n \frac{2n}{3n+\cos(n)} = (-1)^n \frac{2}{3+\cos(n)/n}$ does not go to zero. With the $(-1)^n$ excluded it goes to $\frac{2}{3} \neq 0$, and with the sign included it has no limit at all. Hence the series must not converge at all, by the n^{th} term test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n+1)}{n^2}$

ANSWER: If we take the absolute value of the terms we get $\frac{|\sin(n+1)|}{n^2}$ which is at most $\frac{1}{n^2}$. The series $\sum \frac{1}{n^2}$ converges: You could use the integral test, but it is easiest to note that it is a p-series with $p = 2 > 1$. Hence the series of absolute values of the original series converges, by the comparison test, so the original series converges absolutely.

(c) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

ANSWER: This is the alternating harmonic series. It is an alternating series with terms decreasing to zero, so it converges. But if we take absolute values we get the harmonic series, which diverges. So the original series converges conditionally.

<u>Problem 7</u>

A function $f(x)$ takes on the values given in the table:

x	$f(x)$
3	0
5	5
7	14
9	27
11	44

(a) Use the Trapezoidal Rule to compute an approximation to $\displaystyle\int_3^{11} f(x)\, dx$. You do not need to estimate error.

ANSWER: The given function values are at the dividing points if we take the interval $3 \le x \le 11$ and divide it into four parts, each of width 2. Hence the trapezoidal approximation to the integral is $\frac{2}{2}[f(3) + 2f(5) + 2f(7) + 2f(11) + f(13)] = 0 + 10 + 28 + 54 + 44 = 136$.

(b) Use the Parabolic Rule (Simpson's Rule) to compute an approximation to $\int_3^{11} f(x)\,dx$. You do not need to estimate error.

ANSWER: We use the same division into subintervals: Since the number of subintervals n is 4 which is even, it is appropriate for Simpson's rule. We get $\frac{2}{3}[f(3) + 4f(7) + 2f(9) + 4f(11) + f(13)]$ $= \frac{2}{3}(0 + 20 + 28 + 108 + 44) = \frac{2}{3} \times 200 = \frac{400}{3} = 133.333\ldots$.

Problem 8

Suppose we choose to use the terms of the Maclaurin series for e^x through $\frac{x^4}{4!}$ to approximate the true value of e^x. If we only apply this to x-values between -2 and -1, how accurate will our results be?
ANSWER: We are using $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$ as our approximation to e^x. Those are the terms of the Maclaurin series corresponding to $n = 0,\ 1, \ldots,\ 4$. The remainder term from Taylor's theorem will be R_4, $\frac{f^{(5)}(c)x^5}{5!}$ for some number c between x and 0. For $f(x) = e^x$ all derivatives are just e^x, so $R_4 = \frac{e^c \times x^5}{120}$. We care about the size, the absolute value, of the error. The largest $|x^5|$ can be, for $-2 < x < -1$, is at most $2^5 = 32$. The function e^x is everywhere increasing, so the largest e^c can be for $x \le c \le 0$ is $e^0 = 1$. Hence $|R_4| \le \frac{1 \times 32}{120} = \frac{4}{15} = 0.2666\ldots$. (The actual largest difference between the approximating polynomial and e^x on $[-2, -1]$ occurs at $x = -2$, where the difference is ≈ 0.1979980501, so our bound is "on the correct side" and is actually pretty close.)

Problem 9

(a) Find the radius of convergence and the interval of convergence for the series $\sum_{n=1}^{\infty} (-1)^n \frac{(2x)^n}{n\,3^n}$.

Be sure to show your reasoning.

ANSWER: The ratio $\frac{a_{n+1}}{a_n}$ of the absolute values of successive terms is

$$\frac{\frac{(2|x|)^{n+1}}{(n+1)3^{n+1}}}{\frac{(2|x|)^n}{n\,3^n}} = \frac{2x}{3}\frac{n}{n+1}.$$

Taking the limit as $n \to \infty$ we get $\rho = \frac{2|x|}{3}$. The series will converge (absolutely) when $\rho < 1$, i.e. $-1 < \frac{2x}{3} < 1$, i.e. $-\frac{3}{2} < x < \frac{3}{2}$. Hence the radius of convergence is $\frac{3}{2}$, and the interval of convergence is from $-\frac{3}{2}$ to $\frac{3}{2}$ except that we don't yet know what happens at the end points where $\rho = 1$.

Putting $x = \frac{3}{2}$ into the series yields the alternating harmonic series, which converges, so the series does converge at the right end of the interval. At the left end, $x = -\frac{3}{2}$, we get the harmonic series (the minus signs "cancel out"), which we know diverges. Hence the interval of convergence is $(-\frac{3}{2}, \frac{3}{2}]$.

(b) Does the series $\sum_{n=0}^{\infty} \frac{3^{n+1}}{5^n}$ converge? If your answer is no, tell how you know that. If your answer is yes, tell what the series converges to, i.e.: What is the sum of the series?

ANSWER: You could use the ratio test and get $\rho = \frac{3}{5} < 1$, so the series converges. But the problem asks for the sum of the series and there are relatively few series for which we know the sum. Considering those, this looks like a geometric series. Writing it as $3 + 3 \times \frac{3}{5} + 3 \times (\frac{3}{5})^2 + \cdots$ we see it is a geometric series with first term $a = 3$ and ratio $r = \frac{3}{5}$. Hence the sum is $\frac{a}{1-r} = \frac{3}{1-\frac{3}{5}}$ $= 3 \times \frac{5}{2} = \frac{15}{2}$ or 7.5.