Mathematics 222, Spring 2005 Lecture 4 (Wilson)

First Midterm Exam February 15, 2005 ANSWERS

Notice that some problems can be done in various ways, and I have given only one possible answer. Also remember that answers could be correct despite looking quite different from those given: In particular this applies to indefinite integrals, where the "$+C$" can take many forms. For example, $\sin^{-1}\theta = -\cos^{-1}\theta +$ a constant that could be included in $+C$.

Problem 1

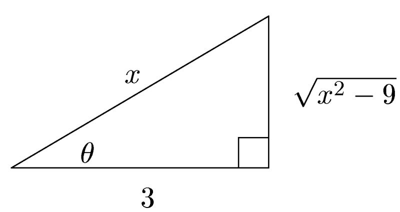
Evaluate the integral: $$\int \sin^2(x)\cos^2(x)\,dx.$$

ANSWER: Use the identities $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ to get $\frac{1}{4}\int(1-\cos 2x)(1+\cos 2x)\,dx$. Multiplying we get $\frac{1}{4}\int(1-\cos^2 2x)\,dx$. Use the second identity again to get $\frac{1}{4}\int(1-\dfrac{1}{2}(1+\cos 4x))\,dx$ which simplifies to $\frac{1}{8}\int dx - \dfrac{1}{8}\int \cos 4x\,dx$, which can then be easily evaluated as $\frac{x}{8} - \frac{1}{32}\sin 4x + C$.

Problem 2

Evaluate the integral: $$\int_3^{3\sqrt{2}} \frac{\sqrt{x^2-9}}{x}\,dx$$

ANSWER: You can either remember a substitution to try or draw a triangle. Using the latter procedure, I draw a right triangle like this:



Then we can read off the relations $x = 3\sec\theta$ and $\sqrt{x^2-9} = 3\tan\theta$. Differentiate x and get $dx = 3\sec\theta\tan\theta\,d\theta$. Now we need to account for the change of variable from x to θ in the limits on the integral: Looking at the picture, when $x = 3$ the hypoteneuse and lower leg are the same length so the triangle collapses, $\theta = 0$. When $x = 3\sqrt{2}$, the triangle becomes an isoceles right triangle (45-45-90 degrees) so $\theta = \frac{\pi}{4}$. Putting these pieces together we work out the integral as

$$\int_0^{\frac{\pi}{4}} \frac{(3\tan\theta)(3\sec\theta\tan\theta)}{3\sec\theta}\,d\theta = \int_0^{\frac{\pi}{4}} 3\tan^2\theta\,d\theta = 3\int_0^{\frac{\pi}{4}}(\sec^2\theta - 1)\,d\theta$$

$$= 3\left[\tan\theta - \theta\right]_0^{\frac{\pi}{4}} = 3(1-\frac{\pi}{4}) = 3 - \frac{3\pi}{4}.$$

Problem 3

Evaluate the integral: $$\int x^2 \sin(x)\, dx$$

ANSWER: We use integration by parts. If we let $u = x^2$ and $dv = \sin(x)\, dx$, then $du = 2x\, dx$ and $v = -\cos(x)$ (plus a constant we ignore in this situation). This gives us

$$\int x^2 \sin(x)\, dx = -x^2 \cos(x) + 2 \int x \cos(x)\, dx.$$

We have to use integration by parts again to evaluate this new integral: Let $u = x$ and $dv = \cos(x)\, dx$, giving $du = dx$ and $v = \sin(x)$, to get

$$\int x^2 \sin(x)\, dx = -x^2 \cos(x) + 2 \int x \cos(x)\, dx = -x^2 \cos(x) + 2 \left[x \sin(x) - \int \sin(x)\, dx \right]$$

$$= -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + C.$$

Problem 4

Evaluate the integral: $$\int \frac{4x^2 + 5x + 3}{(x+1)(x^2+x+1)}\, dx$$

ANSWER: We use a partial fractions decomposition of the fraction that is to be integrated. The degree of the numerator (2) is less than that of the denominator (3), so we are set to proceed. The denominator is already factored: The quadratic factor $x^2 + x + 1$ cannot be factored further since it has no real roots. If you try using the quadratic formula to look for roots you quickly come to $\sqrt{1-4}$ which will be imaginary, not real. With these preliminaries checked we know we can find some numbers A, B, and C, such that

$$\frac{4x^2 + 5x + 3}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 + x + 1}.$$

Multiplying both sides by $(x+1)(x^2+x+1)$ we get $4x^2 + 5x + 3 = A(x^2 + x + 1) + (Bx + C)(x + 1)$. One way to proceed at this point is to extract three equations in the three unknowns A, B, and C by equating the x^2 terms from both sides, the x terms, and the constants. This yields the equations $A + B = 4$, $A + B + C = 5$, and $A + C = 3$, which can be solved as a system of 3 equations in 3 unknowns. We can save a little work by using a different scheme to find A and then using it in those same equations to get B and C: Since the equation $4x^2 + 5x + 3 = A(x^2 + x + 1) + (Bx + C)(x + 1)$ must be true for all values of x, in particular it is true when $x = -1$. When $x = -1$, i.e. $x + 1 = 0$, the last part of the equation disappears, so $4(-1)^2 + 5(-1) + 3 = A((-1)^2 + (-1) + 1)$ which simplifies to $A = 2$. Now using that in the equation $A + B = 4$ we have $B = 2$ and using it in the equation $A + C = 3$ gives $C = 1$. We can rewrite the original problem as

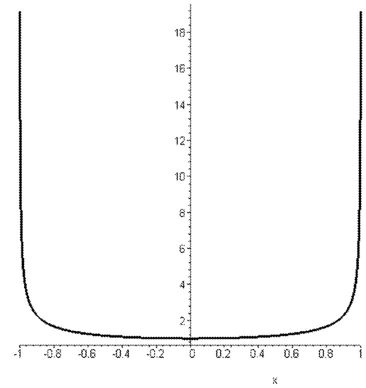
$$\int \frac{4x^2 + 5x + 3}{(x+1)(x^2+x+1)}\, dx = \int \frac{2}{x+1}\, dx + \int \frac{2x+1}{x^2+x+1}\, dx$$

$$= 2 \ln|x+1| + \ln|x^2 + x + 1| + C.$$

Problem 5

At the right is a graph of $\dfrac{1}{\sqrt{1-x^2}}$.

Evaluate the integral $\displaystyle\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$.



ANSWER: We see that the integrand "blows up" at each end of the interval $[-1, 1]$. Hence we need to treat this as an improper integral, using a limit at each end. We can choose any point in the "middle" at which to break this into two integrals: I will use 0.

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \lim_{a\to -1^+} \int_{a}^{0} \frac{dx}{\sqrt{1-x^2}} + \lim_{b\to 1^-} \int_{0}^{b} \frac{dx}{\sqrt{1-x^2}}$$

Now $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$, so we have

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \lim_{a\to -1^+} \int_{a}^{0} \frac{dx}{\sqrt{1-x^2}} + \lim_{b\to 1^-} \int_{0}^{b} \frac{dx}{\sqrt{1-x^2}}$$

$$= \lim_{a\to -1^+} (\arcsin(0) - \arcsin(a)) + \lim_{b\to 1^-} (\arcsin(b) - \arcsin(0)) = (0 - (-\frac{\pi}{2})) + (\frac{\pi}{2} - 0) = \pi.$$

Problem 6

(a) Evaluate the limit $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x}}{4\,\sin(x)}$.

ANSWER: As $x \to 0$, both the numerator and denominator go to 0. Using l'Hôpital's Rule,

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{4\,\sin(x)} = \lim_{x\to 0} \frac{e^x + e^{-x}}{4\,\cos(x)} = \frac{2}{4} = \frac{1}{2}.$$

(b) Evaluate the limit $\displaystyle\lim_{x\to\infty} x^{(2/x^2)}$.

ANSWER: First we need to get $\frac{2}{x^2}$ out of the exponent: Taking the natural logarithm of $x^{(2/x^2)}$ we get $\frac{2}{x^2}\ln(x)$. If we can find the limit of that expression as $x \to \infty$, we can then raise e to the resulting limit and get the answer to the original problem. So now we want $\displaystyle\lim_{x\to\infty} \frac{2\ln x}{x^2}$. Since numerator and denominator both go to ∞, we can apply l'Hôpital's Rule and instead evaluate $\displaystyle\lim_{x\to\infty} \frac{2/x}{2x}$ which gives 0. Hence the answer to the original problem is $e^0 = 1$.

Problem 7

(a) Consider the sequence $a_n = 3 - \frac{2}{n}$.

Give an argument to show that $\{a_n\}$ has a limit. You should either justify this using carefully the definition of the limit of a sequence or by appropriate use of a theorem about sequences from the book.

ANSWER: You could use the definition of the limit of a sequence but it is easier to use the Monotonic Sequence Theorem (Theorem D, page 433) from the text. To use that we need to show the terms of this sequence are non-decreasing and that there is some number U which is bigger than each of the terms. Since $\frac{2}{n}$ is decreasing as n gets larger, and the terms a_n of our sequence have that subtracted from 3, the terms are getting larger as n increases. And each term is less than 3. So the theorem applies and the sequence has a limit. (The theorem says more, that the limit will be less than or equal to 3, and in fact it is 3, but the exam problem did not require this.)

(b) The first several terms of a different sequence $\{a_n\}$ are $1, \frac{2}{2^2-1^2}, \frac{3}{3^2-2^2}, \frac{4}{4^2-3^2}, \ldots$.

(i) Find a formula giving a_n as a formula involving n.

ANSWER: The simplest formula to see is probably $a_n = \dfrac{n}{n^2-(n-1)^2}$.

(ii) This sequence does converge. What is its limit? Show all of your work.

ANSWER: If we take the formula we got in (i) and do some algebra,

$$\frac{n}{n^2-(n-1)^2} = \frac{n}{n^2-n^2+2n-1} = \frac{n}{2n-1}.$$

If we divide numerator and denominator of that last expression by n we get $\frac{1}{2-(1/n)}$. As $n \to \infty$ that clearly goes to $\frac{1}{2-0} = \frac{1}{2}$.

Problem 8

For each series, tell whether it converges or diverges.
If the series converges, tell what its sum is.
Be sure to show your work!

(a) $$\sum_{n=0}^{\infty} \frac{n-1}{n+1}.$$

ANSWER: The limit of the terms $\frac{n-1}{n+1}$ in this series, as $n \to \infty$, is 1. Hence the terms do not go to zero. So the series cannot converge, it must diverge.

(b) $$\sum_{n=2}^{\infty} \left(\frac{2}{3}\right)^n$$

ANSWER: This is a geometric series: each term is $\frac{2}{3}$ times the previous one. The first term is $a = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$, and the ratio $r = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges to

$$\frac{a}{1-r} = \frac{\frac{4}{9}}{1-\frac{2}{3}} = \frac{4}{3}.$$