Final Exam **May 13, 2004**

- **Write your answers to the twelve problems in the spaces provided. If you must continue an answer somewhere other than immediately after the problem statement, be sure (a) to tell where to look for the answer, and (b) to label the answer wherever it winds up. In any case, be sure to circle your final answer to each problem.**
- **On the other side of this sheet there are some facts and formulas and a table for undetermined coefficients.**
- **Wherever applicable, leave your answers in exact forms (using π, e, $\sqrt{3}$, $\ln(2)$, and similar numbers) rather than using decimal approximations.**
- **You may refer to notes you have brought in, as announced in class.**
- **There is scratch paper on the back of the last page.**

BE SURE TO SHOW YOUR WORK, AND EXPLAIN WHAT YOU DID. YOU MAY RECEIVE REDUCED OR ZERO CREDIT FOR UNSUBSTANTIATED ANSWERS. ("I did it on my calculator" and "I used a formula from the book" are not sufficient substantiation...)

Problem	Points	Score
1	14	
2	18	
3	16	
4	18	
5	18	
6	14	
7	16	
8	16	
9	18	
10	20	
11	16	
12	16	
TOTAL	200	

Some formulas, identities, and numeric values you might find useful:

Values of trig functions:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	—

Trig facts:

1. $\tan\theta = \frac{\sin\theta}{\cos\theta}$
2. $\sec\theta = \frac{1}{\cos\theta}$
3. $\sin^2\theta + \cos^2\theta = 1$
4. $\sec^2\theta = \tan^2\theta + 1$
5. $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
6. $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
7. $\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$
8. $\sin^2 x = \frac{1}{2}(1-\cos 2x)$
9. $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

Derivative formulas:

1. $\frac{d}{dx}\tan x = \sec^2 x$
2. $\frac{d}{dx}\sec x = \sec x\,\tan x$
3. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
4. $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
5. $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$
6. $\frac{d}{dx}\ln x = \frac{1}{x}$
7. $\frac{d}{dx}e^x = e^x$

Integral formulas:

1. $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C$, if $n \neq -1$
2. $\int \frac{1}{u}\,du = \ln|u| + C$
3. $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$
4. $\int \frac{du}{1+u^2} = \tan^{-1} u + C$
5. $\int \sec(u)\,du = \ln|\sec(u) + \tan(u)| + C$
6. $\int u\,dv = uv - \int v\,du$

Algebra formulas:

1. $\ln(xy) = \ln(x) + \ln(y)$
2. $a^{x+y} = a^x\,a^y$
3. $a^b = e^{b\ln a}$

Terms to use in y_p, for undetermined coefficients:

For a term in $f(x)$ which is a multiple of	If	Then use a term like
$\sin(kx)$ or $\cos(kx)$	ki is not a root of the characteristic equation	$A\cos(kx) + B\sin(kx)$
	ki is a root of the characteristic equation	$Ax\cos(kx) + Bx\sin(kx)$
e^{nx}	n is not a root of the characteristic equation	Ce^{nx}
	n is a single root of the characteristic equation	$Cx\,e^{nx}$
	n is a double root of the characteristic equation	$Cx^2\,e^{nx}$
A polynomial ax^2+bx+c of degree at most 2	0 is not a root of the characteristic equation	a polynomial Dx^2+Ex+F of the same degree as ax^2+bx+c
	0 is a single root of the characteristic equation	a polynomial Dx^3+Ex^2+Fx of degree one more
	0 is a double root of the characteristic equation	a polynomial $Dx^4+Ex^3+Fx^2$ of degree two more

Problem 1 (14 points)

Set up an integral to compute the area inside one leaf of the 5-leafed rose $r = 3\sin(5\theta)$. Be sure to make clear how you establish the limits of integration.

You do not have to evaluate this integral.

Problem 2 (18 points)
Evaluate the integrals:

(a) $$\int e^{2x} \sin(3x)\, dx$$

(b) $$\int \frac{x^2\, dx}{\sqrt{16 - x^2}}$$

Problem 3 (16 points)

Use Simpson's (Parabolic) Rule to evaluate the integral $\int_1^9 (x^2 + 1)\, dx$, using 4 subintervals.

Problem 4 (18 points)

For the differential equation $y'' - 6y' + 10y = 0$, we can express the set of all solutions as $y = e^{3x}(C_1 \cos x + C_2 \sin x)$. The first part of this problem asks you find that solution. Hence clearly you will get no credit for simply writing down the solution, credit will come from showing how to find the solution, but at the same time you can check your work and also be sure of proceeding to the last part of the problem.

Part I: Find all solutions of $y'' - 6y' + 10y = 0$.

- What is the characteristic (associated) polynomial for this differential equation?

- What are the roots of the characteristic polynomial?

- What role do the roots you found play in writing out the solution $y = e^{3x}(C_1 \cos x + C_2 \sin x)$? Show where each number in the solutions came from as a part of the roots.

Part II: Find all solutions of $y'' - 6y' + 10y = 4e^{4x}$.

Part III: Find the solution of $y'' - 6y' + 10y = 4e^{4x}$ satisfying $y(0) = 5$ and $y'(0) = 16$.

Problem 5 (18 points)

Use Taylor's theorem to estimate the error if we approximate $e^{\frac{1}{2}}$ using the terms of the Maclaurin series for e^x through the term $\frac{x^6}{6!}$. Your estimate should take into account the value $\frac{1}{2}$ at which we are applying this.

You may use the fact $e^{\frac{1}{2}} < 2$.

Problem 6 (14 points)

For the series $\sum_{n=1}^{\infty} \frac{(-3x)^n}{n}$ determine the interval of convergence (convergence set).

Be sure to indicate clearly what happens at any endpoints, and give reasons for your answers.

Problem 7 (16 points)

The equation $25x^2 + 9y^2 = 225$ describes a conic section. For this curve, find:

(a) where it crosses the x-axis, if it does at all (coordinates of point(s))

(b) where it crosses the y-axis, if it does at all (coordinates of point(s))

(c) its focus or foci (coordinates of point(s))

(d) its eccentricity (a number)

Problem 8 (16 points)

Let $\vec{u} = 2\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{v} = 3\vec{j} + 4\vec{k}$ for all parts of this problem.

(a) Find a vector of unit length in the direction of $\vec{v}$.

(b) Find $\cos(\theta)$ where θ is the angle between $\vec{u}$ and $\vec{v}$.

(c) Find the scalar projection of $\vec{u}$ upon $\vec{v}$.

(d) Find the vector projection of $\vec{u}$ upon $\vec{v}$.

Problem 9 (18 points)

(a) Find an equation for the plane which passes through $(2, 5, -6)$ and is parallel to the plane $5x - 3y + 2z = 8$.

(b) Find equations for the line which passes through $(2, 5, -6)$ and is perpendicular to both of the planes in (a). You may express these in parametric or symmetric form.

Problem 10 (20 points)

For each series, tell whether it converges or diverges. Be sure to give reasons for your answers.

(a) $$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

(b) $$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$$

For each series, tell whether it diverges, converges absolutely, or converges conditionally. Be sure to give reasons for your answers.

(c) $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{1000n + 1,000,000}$$

(d) $$\sum_{n=1}^{\infty} (-1)^n \frac{5}{n}$$

Problem 11 (16 points)

Suppose the position of an object is represented by the vector $\vec{r}(t) = \cos(2t)\vec{i} - 3t\vec{j} + \sin(2t)\vec{k}$ at any time t.

(a) Where is the object (coordinates of a point) at time $t = \frac{\pi}{2}$?

(b) What is the velocity (vector) of the object at time $t = \frac{\pi}{2}$?

(c) What is the acceleration (vector) of the object at time $t = \frac{\pi}{2}$?

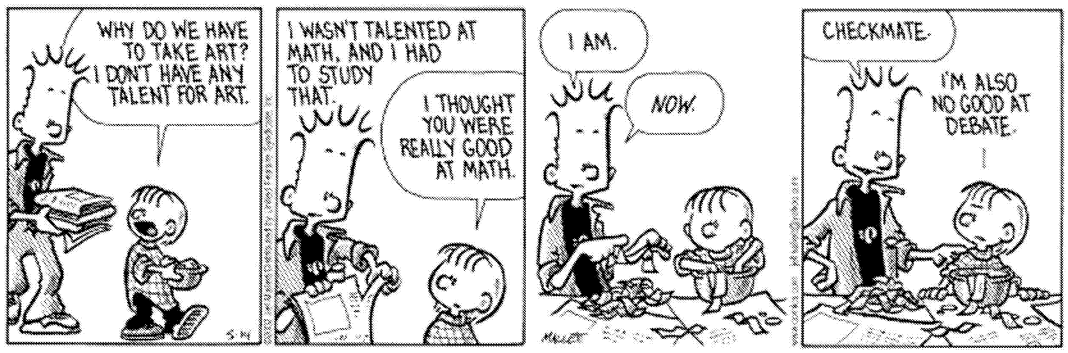
(d) Find equations in symmetric form for the tangent line to the path of this object at $t = \frac{\pi}{2}$.

Problem 12 (16 points)

(a) Evaluate the integral $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx$.

(b) Evaluate the integral $\int_{-1}^{1} \sqrt{1-x^2}\, dx$.

Hint: This is much easier if you use some geometry...



Scratch Paper

(not a command!)