Answers to Second Midterm Exam April 20, 2004

Problem 1 (12 points)
A hyperbola crosses the y-axis at $(0, \pm 12)$, and its foci are at $(0, \pm 13)$.

(a) **Find an equation for the hyperbola.**

Answer: The foci are on the y-axis and symmetric about the origin, so the equation can be written in the form $-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$. The number a will tell where the curve crosses the y-axis, so $a = 12$. The foci will be at $(0, \pm c)$, so $c = 13$, and we know $a^2 + b^2 = c^2$. Thus $144 + b^2 = 169$, so $b^2 = 25$. Now we can write out the equation,

$$-\frac{x^2}{25} + \frac{y^2}{144} = 1.$$

(b) **What is the eccentricity of this hyperbola?**

Answer: The eccentricity is $e = \frac{c}{a} = \frac{13}{12}$.

(c) **What are the asymptotes of this hyperbola? (Give equations for the lines.)**

Answer: If you construct the box centered at the origin and extending horizontally to $\pm b = \pm 5$ and vertically to $\pm a = \pm 12$, the asymptotes are the diagonals of the box. Hence they can be written as $y = \pm \frac{12}{5} x$.

Problem 2 (12 points)
Find the interval of convergence (convergence set) for the series

$$\sum_{n=1}^{\infty} \frac{3\,x^n}{n\,2^n}.$$

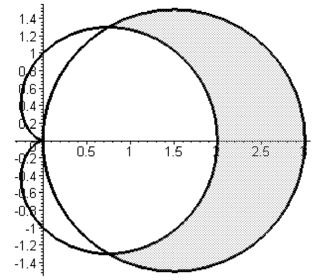
Answer: We set up the ratio test on this series. The ratio of successive terms is

$$\frac{\frac{3x^{n+1}}{(n+1)2^{n+1}}}{\frac{3x^n}{n\,2^n}} = \frac{n}{n+1} \times \frac{x}{2}.$$

The limit of that fraction as $n \to \infty$ is $\frac{x}{2}$ so the series converges (absolutely) when $\left|\frac{x}{2}\right| < 1$ and possibly when when $\left|\frac{x}{2}\right| = 1$. Thus the interval of convergence is at least $-2 < x < 2$ or $(-2, 2)$, and perhaps one or more of the endpoints should be included.
So now we test separately at the endpoints. At $x = 2$ the series becomes $\sum_{n=1}^{\infty} \frac{3\,2^n}{n\,2^n}$ or $\sum_{n=1}^{\infty} \frac{3}{n}$. That is, except for an initial term and a constant multiple, the harmonic series which diverges. Hence our series diverges at $x = 2$. Now try it at $x = -2$. Here the series becomes $\sum_{n=1}^{\infty} \frac{3\,(-2)^n}{n\,2^n}$ or $\sum_{n=1}^{\infty} (-1)^n \frac{3}{n}$. So this time the series is essentially an alternating harmonic series, which converges.
Thus the interval of convergence is $-2 \le x < 2$ or $[-2, 2)$.

Problem 3 (13 points)
Find the area of the region in the plane which is inside the circle $r = 3\cos\theta$ but outside the cardioid $r = 1 + \cos\theta$.

Answer: The curves are shown at the right, with the area we need shaded in. We need to determine where the curves intersect. Setting $3\cos\theta = 1+\cos\theta$ **we get** $\cos\theta = \frac{1}{2}$**, so** $\theta = \pm\frac{\pi}{3}$**. Now we can calculate the area using**



$$\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[(3\cos\theta)^2 - (1+\cos\theta)^2\right] d\theta.$$

Now we need to evaluate that integral. Squaring the two pieces and collecting terms we get

$$\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (8\cos^2\theta - 1 - 2\cos\theta)d\theta = -\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\theta - \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos\theta\, d\theta + 2\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} [1+\cos(2\theta)]\, d\theta.$$

$$= -\frac{\pi}{3} - \sqrt{3} + \frac{4\pi}{3} + \sqrt{3} = \pi.$$

Problem 4 (13 points)

Find the Taylor series for $f(x) = \ln(1+3x)$, **at** $a = \frac{1}{3}$.

Answer: I will construct a table to organize computing the various derivatives and their values at $\frac{1}{3}$**.**

n	$f^{(n)}(x)$	$f^{(n)}(\frac{1}{3})$	$\frac{f^{(n)}(\frac{1}{3})}{n!}$
0	$\ln(1+3x)$	$\ln(2)$	$\ln(2)$
1	$\frac{3}{1+3x}$	$\frac{3}{2}$	$\frac{3}{2}$
2	$-\frac{9}{(1+3x)^2}$	$-\frac{9}{4}$	$-\frac{9}{8}$
3	$\frac{54}{(1+3x)^3}$	$\frac{54}{8}$	$\frac{9}{8}$
4	$-\frac{486}{(1+3x)^4}$	$-\frac{486}{16}$	$-\frac{81}{64}$
...	...	...	...
n	$-\frac{(-3)^n(n-1)!}{(1+3x)^n}$	$(-1)^{n+1}\left(\frac{3}{2}\right)^n (n-1)!$	$(-1)^{n+1}\left(\frac{3}{2}\right)^n \frac{1}{n}$

Now we can write out the series:

$$\ln(2) + \frac{3}{2}\left(x-\frac{1}{3}\right) - \frac{9}{8}\left(x-\frac{1}{3}\right)^2 + \frac{9}{8}\left(x-\frac{1}{3}\right)^3 - \frac{81}{64}\left(x-\frac{1}{3}\right)^4 + \cdots + (-1)^{n+1}\left(\frac{3}{2}\right)^n \frac{1}{n}\left(x-\frac{1}{3}\right)^n + \cdots.$$

Problem 5 (12 points)

(a) **Use the Trapezoidal rule with** $n = 4$ **subintervals to approximate** $\int_1^3 (x^2-1)\,dx$.

Answer: For $n = 4$**, with the interval** $[1,3]$**, we have the width of each subinterval** $h = \frac{3-1}{4} = \frac{1}{2}$**. The division points are (I am writing them all as multiples of** $\frac{1}{2}$ **to make the arithmetic easier later)** $x_0 = \frac{2}{2}$, $x_1 = \frac{3}{2}$, $x_2 = \frac{4}{2}$, $x_3 = \frac{5}{2}$, **and** $x_4 = \frac{6}{2}$**. Then the Trapezoidal rule tells us to compute**

$$\frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right] \quad \textbf{where} f(x) = x^2 - 1.$$

That gives $\frac{1}{4}\left[\left(\frac{2}{2}\right)^2 - 1 + 2\left(\left(\frac{3}{2}\right)^2 - 1\right) + 2\left(\left(\frac{4}{2}\right)^2 - 1\right) + 2\left(\left(\frac{5}{2}\right)^2 - 1\right) + \left(\frac{6}{2}\right)^2 - 1\right]$

or $\frac{1}{4}\left[\left(\frac{4}{4} + \frac{18}{4} + \frac{32}{4} + \frac{50}{4} + \frac{36}{4}\right) - 8\right]$ **which works out to** $6\frac{3}{4}$.

(b) **Calculate exactly** $\int_1^3 (x^2-1)\,dx$.

Answer: $\left(\frac{x^3}{3}-x\right)\Big]_1^3 = (9-3)-(\frac{1}{3}-1) = 6\frac{2}{3}$.

(c) Give an argument based on the shape of the graph of $y = x^2 - 1$ and the nature of the trapezoidal rule to explain why your answer to (b) is larger or smaller than your answer to (a).

Answer: The graph of $y = x^2 - 1$ is everywhere concave up. Hence the straight top edges of the trapezoids whose areas contribute to the calculation in (a) stay above the curve except at their endpoints. That makes the area in the trapezoid greater than the actual area under the curve in each subinterval. Hence the answer from the trapezoidal rule must be greater than the actual integral, for this function, i.e. the answer to (a) should be greater than the answer to (b), and it is.

Problem 6 (13 points)
We want to use a polynomial consisting of some beginning terms of the Maclaurin series for $\sin(x)$ to approximate $\sin(0.2)$. Use the remainder term $R_n(x)$ from Taylor's theorem to decide which terms to use if the error must be at most $0.000001 = 10^{-6}$.
Answer: A formula for the remainder term is $R_n(x) = \frac{f^{(n+1)}(c)\,x^{n+1}}{(n+1)!}$ where $f(x) = \sin(x)$ and c is some number between 0 and 0.2. Depending on n, the $(n+1)^{st}$ derivative of $\sin(x)$ will be $\pm\sin(x)$ or $\pm\cos(x)$, but we don't know which since we don't yet know n. But those functions never get larger than 1 in absolute value, so we can say $|f^{(n+1)}(c)! \leq 1$. This gives us $|R_n(0.2)| \leq \frac{(0.2)^{n+1}}{(n+1)!}$. We can now compute that fraction for different values of n and see what is the smallest value for n that makes it less than or equal to 10^{-6}. For $n = 4$ we get $2.666\ldots \times 10^{-6}$, too big. For $n = 5$ we get $8.888\ldots \times 10^{-8}$, which is small enough, so we use $n = 5$. That means we need the polynomial consisting of the terms from the Maclaurin series for $\sin x$ through the 5^{th} degree term, $x - \frac{x^3}{6} + \frac{x^5}{120}$. That is the answer, you were only asked for the polynomial. The actual approximation would be $(0.2) - \frac{(0.2)^3}{6} + \frac{(0.2)^5}{120} = 0.198669333\ldots$.
For comparison, $\sin(0.2) \approx 0.198669330795\ldots$.

Problem 7 (13 points)

Find all solutions of the differential equation $y'' + 6y' - 13y = 9e^{-x}$.

Answer: First we write the characteristic equation $r^2 + 6r - 13 = 0$: This has roots $r = -3 \pm 2i$.
Now we can write the general solution to the homogeneous equation $y'' + 6y' - 13y = 0$, $y_h = e^{-3x}(C_1 \cos 2x + C_2 \sin 2x)$.
Next we need to find a particular solution y_p to the actual equation $y'' + 6y' - 13y = 9e^{-x}$. Using Undetermined Coefficients we try a solution of the form $y_p = Ce^{-x}$. We need to "plug that into" the equation, so we need its derivatives $y_p' = -Ce^{-x}$ and $y_p'' = Ce^{-x}$. In the left side of the equation this yields $y_p'' + 6y_p' - 13y_p = Ce^{-x} - 6Ce^{-x} - 13Ce^{-x} = -18Ce^{-x}$, which must give $9e^{-x}$, so $C = -\frac{1}{2}$. Thus we get $y_p = -\frac{1}{2}e^{-x}$.
Combining, we get the general solution $y(x) = -\frac{1}{2}e^{-x} + e^{-3x}(C_1 \cos 2x + C_2 \sin 2x)$.

Problem 8 (12 points)
An ellipse is parametrized as $x = 2\sin(t)$ and $y = 4\cos(t)$. Find an equation for the tangent line to this curve at the point where $t = \frac{\pi}{4}$.

Answer: We need the coordinates (x_0, y_0) **of the point on the curve where** $t = \frac{\pi}{4}$**, and the derivative** $\frac{dy}{dx}$ **at that point to determine the slope. The point on the curve has** x **coordinate** $2\sin\frac{\pi}{4} = 2\frac{\sqrt{2}}{2} = \sqrt{2}$ **and** y **coordinate** $4\cos\frac{\pi}{4} = 4\frac{\sqrt{2}}{2} = 2\sqrt{2}$**.**

The slope is $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-4\sin t}{2\cos t}.$$

At $t = \frac{\pi}{4}$ **that gives** -2**.**
Hence the tangent line goes through $(\sqrt{2}, 2\sqrt{2})$ **with slope** -2**. Now we can write the equation,** $y - 2\sqrt{2} = -2(x - \sqrt{2})$ **or** $y = -2x + 4\sqrt{2}$**.**