Mathematics 222, Lecture 2 (Wilson) Your Name: ____________________

Final Exam May 15, 2002

- **Write your answers to the eleven problems in the spaces provided. If you must continue an answer somewhere other than immediately after the problem statement, be sure (a) to tell where to look for the answer, and (b) to label the answer wherever it winds up. In any case, be sure to circle your final answer to each problem.**
- **On the other side of this sheet there is a collection of facts and formulas.**
- **Wherever applicable, leave your answers in exact forms (using π, e, $\sqrt{3}$, $\ln(2)$, and similar numbers) rather than using decimal approximations.**
- **You may refer to notes you have brought in on one sheet of paper or two index cards, as announced in class.**
- **There is scratch paper on the back of the last page.**

BE SURE TO SHOW YOUR WORK, AND EXPLAIN WHAT YOU DID. YOU MAY RECEIVE REDUCED OR ZERO CREDIT FOR UNSUBSTANTIATED ANSWERS. ("I did it on my calculator" and "I used a formula from the book" are not sufficient substantiation...)

Problem	Points	Score
1	16	
2	21	
3	20	
4	20	
5	17	
6	18	
7	16	
8	16	
9	20	
10	20	
11	16	
TOTAL	200	

Some formulas, identities, and numeric values you might find useful:

Values of trig functions:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	—

Trig facts:

1. $\tan\theta = \frac{\sin\theta}{\cos\theta}$
2. $\sec\theta = \frac{1}{\cos\theta}$
3. $\sin^2\theta + \cos^2\theta = 1$
4. $\sec^2\theta = \tan^2\theta + 1$
5. $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
6. $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
7. $\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$
8. $\sin^2 x = \frac{1}{2}(1-\cos 2x)$
9. $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

Derivative formulas:

1. $\frac{d}{dx}\tan x = \sec^2 x$
2. $\frac{d}{dx}\sec x = \sec x\,\tan x$
3. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
4. $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
5. $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$
6. $\frac{d}{dx}\ln x = \frac{1}{x}$
7. $\frac{d}{dx}e^x = e^x$

Integral formulas:

1. $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C$, if $n \neq -1$
2. $\int \frac{1}{u}\,du = \ln|u| + C$
3. $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$
4. $\int \frac{du}{1+u^2} = \tan^{-1} u + C$
5. $\int \sec(u)\,du = \ln|\sec(u) + \tan(u)| + C$
6. $\int u\,dv = uv - \int v\,du$

Algebra formulas:

1. $\ln(xy) = \ln(x) + \ln(y)$
2. $a^{x+y} = a^x\,a^y$
3. $a^b = e^{b\ln a}$

Problem 1 (16 points)
Let $A = (0, 2, 0)$, $B = (1, 0, 0)$, and $C = (0, 0, 3)$.

(a) Use vector methods to find the area of the triangle whose vertices are A, B, and C.

(b) Find an equation for the plane through points A, B, and C.

Evaluate the integrals:

(a) $$\int_0^{\frac{1}{2}} \frac{4x^2}{\sqrt{1-x^2}}\,dx$$

(b) $$\int 4x\,\sec^2 2x\,dx$$

(c) $$\int_0^1 \frac{1}{x^{0.99}}\,dx$$

Problem 3 (20 points)
Solve the initial value problem:

$$y' + 2xy = e^{x-x^2}, \qquad y(0) = 3.$$

(a) Write out the first five terms of the Maclaurin series for $f(x) = e^{3x}$.

(b) Calculate (and justify) a bound on the error you would get if you used your answer in (a) to approximate $e^{0.3}$.

(Be careful what error estimate you use and how you choose x.)

Problem 5 (17 points)
Let $\vec{u} = \vec{i} + \vec{j}$ and $\vec{v} = \vec{i} - \vec{k}$.

(a) Find equations for the line through $(1, -1, 2)$ in the direction of $\vec{u}$.

(b) Find equations for the line through $(1, -1, 2)$ in the direction of $\vec{v}$.

(c) What is the angle between the lines found in (a) and (b)?

Problem 6 (18 points)

(a) Does the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

converge or diverge? (Be sure to give reasons!)

(b) Does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1}$$

converge absolutely, converge conditionally, or diverge? (Be sure to give reasons!)

(c) Does the series

$$2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \cdots = \sum_{n=0}^{\infty} 2\left(-\frac{1}{4}\right)^n$$

converge? (Be sure to give reasons!) If it converges, what is its sum?

Problem 7 (16 points)
Find a parametric representation for motion along the curve $x = 2y^2 - 1$ with parameter t such that $t = 0$ corresponds to the point $(-1, 0)$ on the curve and $t = 2$ corresponds to the point $(1, 1)$.

Problem 8 (16 points)
Write the equation

$$16x^2 - 64x - 9y^2 + 18y + 199 = 0$$

in the form of an equation for a conic section in standard form, for some coordinates x' and y'. (I.e., with respect to the x' and y' axes, the center should be at the origin if this is an ellipse or hyperbola and the vertex should be at the origin if it is a parabola.)
Using your rewritten equation, what are the (x, y) (original) coordinates of:

(if it is an ellipse) The foci and the ends of the major and minor axes?

(if it is a parabola) The focus and directrix (as an equation in x and y)?

(if it is a hyperbola) The foci and asymptotes (as equations in x and y)?

Problem 9 (20 points)
Find all solutions of

$$y'' - 4y' + 13y = 26x - 21.$$

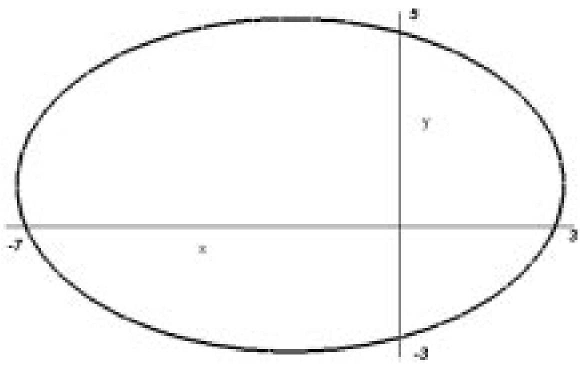
Problem 10 (20 points)

Find the circumference (the arc length once around) of the cardioid $r = \cos\theta - 1$.

(You should come up with an integral that you can evaluate using the formulae provided earlier in this exam.)

Problem 11 (16 points)

An ellipse with its major axis horizontal extends from $x = -7$ to $x = 3$ and from $y = -3$ to $y = 5$. Find an equation for this ellipse. What is its eccentricity?
(The labels on the drawing show the extreme points on the curve, not the places where it crosses the axes.)



Scratch Paper