Final Exam May 15, 2002

Problem 1 (16 points)
Let $A = (0, 2, 0)$, $B = (1, 0, 0)$, and $C = (0, 0, 3)$.

(a) Use vector methods to find the area of the triangle whose vertices are A, B, and C.

Answer: We construct the vectors $\overrightarrow{AB} = \vec{\imath} - 2\vec{\jmath}$ and $\overrightarrow{AC} = -2\vec{\jmath} + 3\vec{k}$. (Any other pairs would work so long as they were not just the same points used twice.) The area we want will be $\frac{1}{2}$ of the length of the cross product of those vectors, $\frac{1}{2}|-6\vec{\imath} - 3\vec{\jmath} - 2\vec{k}| = \frac{1}{2}\sqrt{36 + 9 + 4} = \frac{7}{2}$.

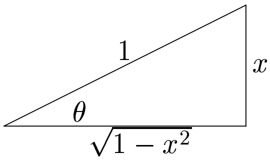
(b) Find an equation for the plane through points A, B, and C.

Answer: The cross product we found in (a) gives us a vector $\perp$ the plane, and we can use A or either of the other two points as a point on the plane. Using A we get $-6(x-0) - 3(y-2) - 2(z-0) = 0$ which we can rewrite as $6x + 3y + 2z = 6$.

Problem 2 (21 points)
Evaluate the integrals:

(a) $$\int_0^{\frac{1}{2}} \frac{4x^2}{\sqrt{1-x^2}}\, dx$$

Answer: If we label a right triangle this way,



we have $x = \sin\theta$, $dx = \cos\theta\, d\theta$, and $\sqrt{1-x^2} = \cos\theta$. When $x = 0$ the triangle collapses and $\theta = 0$. When $x = \frac{1}{2}$ (and $\sqrt{1-x^2} = \frac{\sqrt{3}}{2}$) the triangle is a "30-60-90" triangle so $\theta = \frac{\pi}{6}$. Hence the integral can be rewritten as

$$\int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta\cos\theta\, d\theta}{\cos\theta}$$

and we immediately cancel the $\cos\theta$ factors (non-zero in this range of x values) to get

$$4\int_0^{\frac{\pi}{6}} \sin^2\theta\, d\theta.$$

Using an identity for $\sin^2\theta$ gives

$$\int_0^{\frac{\pi}{6}} (1 - \cos\theta) d\theta.$$

This gives

$$[2\theta - \sin 2\theta]_0^{\frac{\pi}{6}} = 2\frac{\pi}{6} - \sin\frac{\pi}{3} = \frac{\pi}{3} - \frac{\sqrt{3}}{2}.$$

(b) $$\int 4x \sec^2 2x \, dx$$

Answer: We use integration by parts, letting $u = 4x$ and $dv = \sec^2 2x \, dx$. Then $du = 4dx$ and $v = \frac{1}{2}\tan 2x$, so the integral becomes

$$2x \tan 2x - 2\int \tan 2x \, dx = 2x \tan 2x - 2\int \frac{\sin 2x}{\cos 2x} \, dx.$$

We use the substitution $u = \cos 2x$ and $du = -2\sin 2x \, dx$ for this latter integral and get as the answer

$$2x \tan 2x + \ln|\cos 2x| + C.$$

(c) $$\int_0^1 \frac{1}{x^{0.99}} \, dx$$

Answer: We rewrite this improper integral as a limit, and also rearrange the power of x, getting

$$\lim_{a\to 0+} \int_a^1 x^{-0.99} \, dx = \lim_{a\to 0+} \left[\frac{x^{0.01}}{0.01}\right]_a^1 = \lim_{a\to 0+} \left[\frac{1^{0.01}}{0.01} - \frac{a^{0.01}}{0.01}\right] = 100.$$

Problem 3 (20 points)
Solve the initial value problem:

$$y' + 2xy = e^{x-x^2}, \qquad y(0) = 3.$$

Answer: The equation is first order linear: In the notation of the textbook, $P(x) = 2x$ and $Q(x) = e^{x-x^2}$. Integrating $P(x)$ we get x^2 so we use as an integrating factor $\rho = e^{x^2}$. The general solution to the differential equation is $y = \frac{1}{\rho}\int \rho Q(x) \, dx = e^{-x^2}\int e^{x^2} e^{x-x^2} \, dx = e^{-x^2}\int e^x \, dx = e^{-x^2}[e^x + C] = e^{x-x^2} + Ce^{-x^2}$. Using the given initial condition, $3 = 1 + C$ so $C = 2$. Thus the solution is $y(x) = e^{x-x^2} + 2e^{-x^2}$.

Problem 4 (20 points)

(a) Write out the first five terms of the Maclaurin series for $f(x) = e^{3x}$.

Answer: We can use the known Maclaurin series for e^x and substitute $3x$ for x, getting

$$e^{3x} = 1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{6} + \frac{81x^4}{24} + \cdots$$

to show for the first five terms.

(b) Calculate (and justify) a bound on the error you would get if you used your answer in (a) to approximate $e^{0.3}$.

Answer: To use those terms to approximate $e^{0.3}$ we will want to substitute $x = 0.1$. The Lagrange form of the remainder term from Taylor's theorem tells us the error is at most $|\frac{f^{(5)}(c)x^5}{5!}|$, where $f^{(5)}(c)$ denotes the fifth derivative of the function $f(x) = e^{3x}$ evaluated at a number c between 0 and x. Hence $f^{(5)}(c) = 243e^{3c}$, with $0 < c < 0.1$. Since the graph of e^{3c} is climbing toward the right, the largest this can be is $e^{0.3}$. You could use various estimates of how large this is: perhaps the simplest is to say that $e^{0.3} < e^1 = e < 3$. Thus $f^{(5)}(c) < 243 \times 3 = 729$, and the error is less than $\frac{729\times(0.1)^5}{5!} \approx 6 \times 10^{-5}$.

Problem 5 (17 points)
Let $\vec{u} = \vec{i} + \vec{j}$ and $\vec{v} = \vec{i} - \vec{k}$.

(a) Find equations for the line through $(1, -1, 2)$ in the direction of $\vec{u}$.

Answer: In parametric form: $x = 1 + t$, $y = -1 + t$, and $z = 2$. In symmetric or Cartesian form: $\frac{x-1}{1} = \frac{y+1}{1} = \frac{z-2}{0}$.

(b) Find equations for the line through $(1, -1, 2)$ in the direction of $\vec{v}$.

Answer: In parametric form: $x = 1 + t$, $y = -1$, and $z = 2 - t$. In symmetric or Cartesian form: $\frac{x-1}{1} = \frac{y+1}{0} = \frac{z-2}{-1}$.

(c) What is the angle between the lines found in (a) and (b)?

Answer: If we call the angle θ, $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$ so $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$. Thus $\theta = \frac{\pi}{3}$ or 60 degrees.

Problem 6 (18 points)

(a) Does the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

converge or diverge? (Be sure to give reasons!)

Answer: Note that $\frac{1}{n^2+1} < \frac{1}{n^2}$. The series $\sum \frac{1}{n^2}$ is a p-series with $p = 2 > 1$ and so it converges. Hence the original series converges by the comparison test.

(b) Does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 + 1}$$

converge absolutely, converge conditionally, or diverge? (Be sure to give reasons!)

Answer: Since the terms of the series in (a) are just the absolute values of the terms in this series, and series (a) converged, this series converges absolutely.

(c) Does the series

$$2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \cdots = \sum_{n=0}^{\infty} 2\left(-\frac{1}{4}\right)^n$$

converge? (Be sure to give reasons!) If it converges, what is its sum?

Answer: This is a geometric series with first term $a = 2$ and ratio $r = -\frac{1}{4}$. Since $|r| < 1$, the series converges, and it converges to $\frac{a}{1-r} = \frac{2}{1+\frac{1}{4}} = \frac{8}{5}$.

Problem 7 (16 points)
Find a parametric representation for motion along the curve $x = 2y^2 - 1$ with parameter t such that $t = 0$ corresponds to the point $(-1, 0)$ on the curve and $t = 2$ corresponds to the point $(1, 1)$.
Answer: Since the equation is given already solved for one variable, x in terms of y, we can just let the parameter be y to start with: Let $y = t$ and $x = 2t^2 - 1$. This is bound to give the correct set of points, but does it meet the criteria for specific t values? At $t = 0$ this gives $y = 0$ and $x = -1$, the point $(-1, 0)$, as

requested. But at $t=2$ this parametrization gives $y=2$ and $x=7$, which is not the point $(1,1)$ we had to get. But we note that at $t=1$ we do go through the correct point, we just "get there too soon." So we shift the scale on t, replacing any particular value of t by half as much. Let $y=\frac{t}{2}$ and $x=2\left(\frac{t}{2}\right)^2-1=\frac{t^2}{2}-1$. This still gives $(-1,0)$ when $t=0$, and now we get at $t=2$ the point $(1,1)$ as required. In summary: Use the parametrization $x=\frac{t^2}{2}-1$ and $y=\frac{t}{2}$.

Problem 8 (16 points)
Write the equation

$$16x^2-64x-9y^2+18y+199=0$$

in the form of an equation for a conic section in standard form, for some coordinates x' and y'. (I.e., with respect to the x' and y' axes, the center should be at the origin if this is an ellipse or hyperbola and the vertex should be at the origin if it is a parabola.)
Using your rewritten equation, what are the (x,y) (original) coordinates of:

(if it is an ellipse) The foci and the ends of the major and minor axes?

(if it is a parabola) The focus and directrix (as an equation in x and y)?

(if it is a hyperbola) The foci and asymptotes (as equations in x and y)?

Answer: First complete the squares:

$$16(x^2-4x\quad)-9(y^2-2y\quad)=-199$$

$$16(x^2-4x+4)-9(y^2-2y+1)=-199+64-9=-144$$

Divide by -144 to get

$$-\frac{(x-2)^2}{9}+\frac{(y-1)^2}{16}=1 \text{ or } -\frac{(x')^2}{3^2}+\frac{(y')^2}{4^2}=1$$

where $x'=x-2$ and $y'=y-1$. This is a hyperbole, opening up and down along the y' axis. Its center is where $x'=0$ and $y'=0$, i.e. $x=2$ and $y=1$ on the original axes. The foci will be at $\pm c$ along the y' axis, where $c^2=9+16$ so $c=5$. Hence the foci are at $(0,\pm 5)$ on the (x',y') axes. Using $x=x'+2$ and $y=y'+1$ we get that the foci are the points $(2,6)$ and $(2,-4)$ on the original (x,y) axes.
The asymptotes are $y'=\pm\frac{4}{3}x'$. The line $y'=\frac{4}{3}x'$ converts to the original axes as $y-1=\frac{4}{3}(x-2)$ or $y=\frac{4}{3}x-\frac{5}{3}$. The other asymptote, $y'=-\frac{4}{3}x'$, converts to $y=-\frac{4}{3}x+\frac{11}{3}$.

Problem 9 (20 points)
Find all solutions of

$$y''-4y'+13y=26x-21.$$

Answer: First we find the general solution y_h of $y''-4y'+13y=0$. The characteristic equation $r^2-4r+13=0$ has roots $r=\frac{4\pm\sqrt{16-52}}{2}=2\pm 3i$. Hence $y_h=e^{2x}(C_1\cos 3x+C_2\sin 3x)$.
Now we find a particular solution y_p to the equation given, $y''-4y'+13y=26x-21$, using the method of undetermined coefficients. We should be able to find a solution y_p that is a polynomial of the same degree as $26x-21$, so we let $y_p=Ax+B$ for some A and B. Then $y_p'=A$ and $y_p''=0$. Putting these in the equation we get $0-4A+13(Ax+B)=26x-21$. Equating the constants gives $-4A+13B=-21$ and from the x terms we get $13A=26$. Thus we have $A=2$ and $B=-1$, so the solution is $y_p=2x-1$.
Finally, we combine these to get $y(x)=e^{2x}(C_1\cos 3x+C_2\sin 3x)+2x-1$ as the general solution to the equation.

Problem 10 (20 points)
Find the circumference (the arc length once around) of the cardioid $r = \cos\theta - 1$.
(You should come up with an integral that you can evaluate using the formulae provided earlier in this exam.)
Answer: This cardioid is symmetric about the x-axis, with the "notch" on the left side. Letting θ go from 0 to 2π takes us once around the cardioid. Hence we can compute the circumference as

$$\int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(\cos\theta - 1)^2 + (-\sin\theta)^2}\, d\theta$$

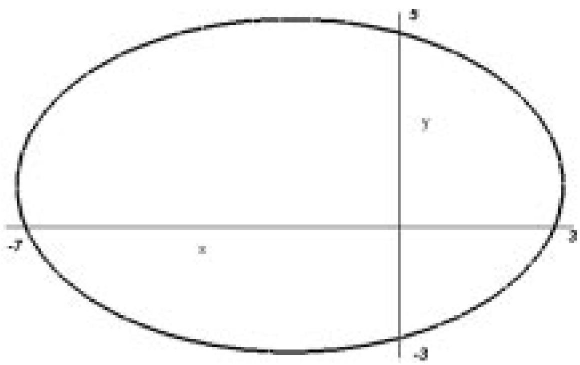
$$= \int_0^{2\pi} \sqrt{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta.$$

We use the formula $\sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta)$, so $2 - 2\cos\theta = 4\sin^2\frac{\theta}{2}$, to get

$$\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta = \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta = 2\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta = -4\left[\cos\frac{\theta}{2}\right]_0^{2\pi} = -4\cos\pi + 4\cos 0 = 8.$$

Problem 11 (16 points)

An ellipse with its major axis horizontal extends from $x = -7$ to $x = 3$ and from $y = -3$ to $y = 5$. Find an equation for this ellipse. What is its eccentricity?
(The labels on the drawing show the extreme points on the curve, not the places where it crosses the axes.)



Answer: The center of the ellipse has its x-coordinate half-way between -7 and $+3$, at $x = -2$, and its y-coordinate half-way between -3 and $+5$, at $y = 1$. Thus on axes for x' and y', with $x' = x + 2$ and $y' = y - 1$, the center of the ellipse is at the origin. The major axis is horizontal so the equation is of the form $\frac{(x')^2}{a^2} + \frac{(y')^2}{b^2} = 1$. The curve crosses the x'-axis at $x' = \pm 5$ and the y'-axis at $y' = \pm 4$, so $a = 5$ and $b = 4$. Hence we can write the equation as

$$\frac{(x+2)^2}{25} + \frac{(y-1)^2}{16} = 1.$$

This is an acceptable form, or you can rewrite the equation in various ways such as $16x^2 + 25y^2 + 64x - 50y - 311 = 0$.
The eccentricity of the ellipse is $\frac{c}{a} = \frac{\sqrt{25-16}}{5} = \frac{3}{5}$.