Exam 1: February 21, 2002 Answers to Shared Portion of Exam

Problem 1 (16 points)

Evaluate the integrals:

(a)
$$\int \frac{dx}{x^2\sqrt{4+x^2}}$$

Draw a right triangle with an acute angle labelled θ, the leg opposite θ labelled x, the leg adjacent to θ labelled 2, and the hypoteneuse labelled $\sqrt{4+x^2}$. Then $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$, and $\sqrt{4+x^2} = 2\sec\theta$. Hence $\int \frac{dx}{x^2\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{(4\tan^2\theta)(2\sec\theta)} = \frac{1}{4}\int \frac{\cos\theta\, d\theta}{\sin^2\theta} = -\frac{1}{4}\frac{1}{\sin\theta} + C = -\frac{1}{4}\frac{\sqrt{4+x^2}}{x} + C$.

(b)
$$\int \arctan(x)\, dx$$

Use parts: Let $u = \arctan x$, $dv = dx$, so $v = x$ and $du = \frac{1}{1+x^2}dx$. Then the integral becomes $x\arctan(x) - \int \frac{x}{1+x^2}dx$ and if we substitute for $1+x^2$ we get $x\arctan(x) - \frac{1}{2}\ln|1+x^2| + C$.

Problem 2 (16 points)

Evaluate the integrals. If you claim an integral does not exist be sure to give reasons!

(a)
$$\int_0^4 \frac{x}{x^2-1}\, dx$$

You can show that this diverges using a comparison. Instead I will work it out completely: Since the denominator vanishes at $x = 1$ this is an improper integral. We break it into two pieces, $\int_0^1 \frac{x}{x^2-1}\, dx$ and $\int_1^4 \frac{x}{x^2-1}\, dx$. Each of these needs to be expressed as a limit,

$$\lim_{b\to 1^-} \int_0^b \frac{x}{x^2-1}\, dx + \lim_{a\to 1^+} \int_a^4 \frac{x}{x^2-1}\, dx.$$

Now $\int \frac{x}{x^2-1}\, dx = \frac{1}{2}\ln|x^2-1| + C$, so we have

$$\lim_{b\to 1^-} (\frac{1}{2}\ln|b^2-1| - \frac{1}{2}\ln|-1|) + \lim_{a\to 1^+} (\frac{1}{2}\ln|15| - \ln|a^2-1|.$$

As $c \to 1$ from either side, $c^2-1>0$ so $|c^2-1| \to 0$ from above and $\ln|c^2-1| \to -\infty$. Thus each piece and the whole integral diverge.

(b)
$$\int_0^{\frac{\pi}{3}} \tan^3 x\, dx$$

We use $\sec^2 x = \tan^2 x + 1$ to rewrite the integral as $\int(\sec^2 x - 1)\tan x\, dx = \int \sec x(\sec x \tan x)dx - \int \tan x\, dx = \int \sec x(\sec x\tan x)dx - \int \frac{\sin x}{\cos x}dx = \frac{\sec^2 x}{2} + \ln|\cos x| + C$. In order to evaluate this at the end points on the integral, recall that $\cos 0 = 1$ so $\sec 0 = 1$ and $\cos\frac{\pi}{3} = \frac{1}{2}$ so $\sec\frac{\pi}{3} = 2$. Then the definite integral gives $(\frac{1}{2}\sec^2\frac{\pi}{3} + \ln|\cos\frac{\pi}{3}|) - (\frac{1}{2}\sec^2 0 + \ln|\cos 0|) = \frac{1}{2}(4-1) + (\ln\frac{1}{2} - \ln 1) = \frac{3}{2} - \ln 2$.

Problem 3 (16 points)
Evaluate the integrals. If you claim an integral does not exist be sure to give reasons!

(a) $$\int_1^4 \frac{\ln x}{\sqrt{x}}\,dx$$

We use integration by parts: Let $u = \ln x$, $dv = x^{-\frac{1}{2}}dx$, $du = \frac{1}{x}dx$, and $v = 2\sqrt{x}$. Then the corresponding indefinite integral becomes $2\sqrt{x}\,\ln x - 2\int \frac{\sqrt{x}}{x}dx = 2\sqrt{x}\,\ln x - 2\int x^{-\frac{1}{2}}dx = 2\sqrt{x}\,\ln x - 4\sqrt{x} + C$. Hence the original problem becomes $(2\sqrt{4}\,\ln 4 - 4\sqrt{4}) - (2\sqrt{1}\,\ln 1 - 4\sqrt{1}) = 4\ln 4 - 4$ or $8\ln 2 - 4$.

(b) $$\int_e^\infty \frac{dx}{x(\ln x)^3}$$

This is improper so we rewrite it as

$$\lim_{b\to\infty}\int_e^b \frac{dx}{x(\ln x)^3} \text{ and using } u = \ln x \text{ we get } \lim_{b\to\infty}\left[-\frac{1}{2}\frac{1}{(\ln x)^2}\right]_e^b.$$

That gives

$$\lim_{b\to\infty}\left(-\frac{1}{2(\ln b)^2}\right) + \frac{1}{2}\frac{1}{(\ln e)^2} = 0 + \frac{1}{2} = \frac{1}{2}.$$

Problem 4 (16 points)
Find the limits of the sequences:

(a) The sequence of partial sums of the series $\displaystyle\sum_{k=0}^{\infty}\frac{4^k}{3^{2k-1}}$

There are two ways to approach this: You can realize that the limit of the sequence of partial sums is by definition the sum of the series, and find the sum of the series by some means, or you can figure out what the sequence is and find its limit. Since the series is geometric, either way works easily. Here we work it out the second way. The k^{th} term in the series can be written as $3 \times \frac{4^k}{3^{2k}} = 3 \times \frac{4^k}{9^k}$. Since this is geometric, the sum of its first n terms would be $a\left(\frac{1-r^{n+1}}{1-r}\right)$, if we let a be the first term and r be the ratio between terms, and for this series $a = 3$ and $r = \frac{4}{9}$. Hence the sequence we are asked about has for its n^{th} term

$$3\left(\frac{1-(\frac{4}{9})^n}{1-\frac{4}{9}}\right).$$

Since $\frac{4}{9} < 1$, the numerator goes to 1. Thus the limit of the sequence is $\frac{3}{1-\frac{4}{9}} = 3 \times \frac{9}{5} = \frac{27}{5}$.

(b) $$a_n = n\,\sin\left(\frac{1}{n}\right)$$

As $n \to \infty$, $\frac{1}{n} \to 0$ and so $\sin\frac{1}{n} \to 0$. Thus the members of the sequence are a product of something going to ∞ and something going to 0. That is frequently a good time to use l'Hôpital's rule: Considering this as a function on the real numbers and not just the integers,

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}$$

which goes to $\frac{0}{0}$ so we can use l'Hôpital's rule. Taking derivatives of numerator and denominator we get

$$\lim_{x\to\infty} \frac{-\frac{1}{x^2}\cos\frac{1}{x}}{-\frac{1}{x^2}} = \cos 0 = 1.$$

Answers to the part of the exam for just Wilson's section:
Problem 1 (12 points)

(a-1) Find an expression for the sum s_n of the first n terms of the series

$$\frac{2}{2\cdot 3} + \frac{2}{3\cdot 4} + \frac{2}{4\cdot 5} + \cdots = \sum_{n=2}^{\infty} \frac{2}{n(n+1)}.$$

The sum of the first n terms of the series is $2\left(\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n+1)(n+2)}\right)$ and we can rewrite each of the fractions, $\frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3}$, $\frac{1}{3\cdot 4} = \frac{1}{3} - \frac{1}{4}$, $\frac{1}{4\cdot 5} = \frac{1}{4} - \frac{1}{5}$, ..., $\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$. All terms except the first and last cancel out and we are left with $2\left(\frac{1}{2} - \frac{1}{n+2}\right)$.

(a-2) What is the sum of this series?
As $n \to \infty$ the n^{th} partial sum $2\left(\frac{1}{2} - \frac{1}{n+2}\right)$ goes to $2\left(\frac{1}{2}\right) = 1$.

(b) The series

$$\sum_{n=1}^{\infty} (-1)^n \frac{3}{2n+1}$$

does converge. If we compute the sum of just the first 5 terms:
(i) How much could this sum differ from the sum of the entire series?
(ii) Would the sum of the first 5 terms be larger or smaller than the sum of the series?

This is an alternating series whose terms are strictly decreasing in size. Hence we can estimate the remainder after summing n terms by looking at the $(n+1)^{st}$ term. The first few terms of the series are $-\frac{3}{3} + \frac{3}{5} - \frac{3}{7} + \frac{3}{9} - \frac{3}{11} + \frac{3}{13} - \frac{3}{15} + \ldots$ so if we sum just the first five terms then the first term omitted is $+\frac{3}{13}$. Hence the difference between the sum of the first five terms and the sum of the whole series is at most $\frac{3}{13}$. Since the first term omitted is positive, the sum of the first five terms is $\leq$ the sum of the whole series.

Problem 2 (12 points)
For each of these series, tell whether it converges. Regardless of whether your answer is "converges" or "diverges," be sure to give reasons for your answer!

(a) $$\sum_{n=1}^{\infty} \frac{n^5}{5^n}$$

Use the ratio test: The ratio

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^5)}{5^{n+1}}}{\frac{n^5}{5^n}} = \frac{(n+1)^5}{n^5}\frac{5^n}{5^{n+1}} = \frac{1}{5}\frac{(n+1)^5}{n^5}.$$

The limit as $n \to \infty$ of that ratio is $\rho = 1 \times \frac{1}{5} = \frac{1}{5} < 1$ so the series converges.

(b) $$\sum_{n=1}^{\infty} \frac{1}{n^3+2}$$

Note that $\frac{1}{n^3+2} < \frac{1}{n^3}$, and $\sum \frac{1}{n^3}$ converges (p-series, $p = 3 > 1$), so the series converges by the comparison test.

(c) $$\sum_{n=1}^{\infty} \frac{2}{n+\ln n}$$

Note that $\ln n < n$ so $\frac{2}{n+\ln n} > \frac{2}{n+n} = \frac{1}{n}$. Hence the series diverges by comparison to the harmonic series.

Problem 3 (12 points)

For each of these series, tell whether it converges absolutely, conditionally or not at all. No matter what the rest of your answer, be sure to give reasons!

(a) $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2n^2-1}$$

The terms of this alternating series are decreasing in size and have limit 0. Hence the series converges by Leibniz' theorem.

But if we make all the signs positive the terms become $\frac{n}{2n^2-1} > \frac{n}{2n^2} = \frac{1}{2}\frac{1}{n}$. Since the harmonic series diverges, this series diverges. Hence the original series does not converge absolutely.

Since the series converges but not absolutely, it converges conditionally.

(b) $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2n-1}$$

The terms $\frac{n}{2n-1}$ of this series approach $\frac{1}{2}$ as $n \to \infty$, not zero. Hence the series cannot converge at all.

(c) $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n)}{n^2}$$

Consider the series after taking absolute values of all terms: The terms are then $\frac{|\sin n|}{n^2}$. Since $|\sin n| \leq 1$, these terms are $\leq \frac{1}{n^2}$. The series $\sum \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$) so the series of absolute values of our series converges by comparison to this convergent series. Hence the series converges absolutely. (And hence it converges.)