Mathematics 222, Spring 2007 Lecture 3 (Wilson)

Final Exam May 17, 2007 ANSWERS

Problem 1 (15 points)

For each of the following equations, indicate its graph by filling in the blank with the number from the picture below.

ANSWERS FILLED IN:

(a) 4 $r = 1 + \sin\theta$

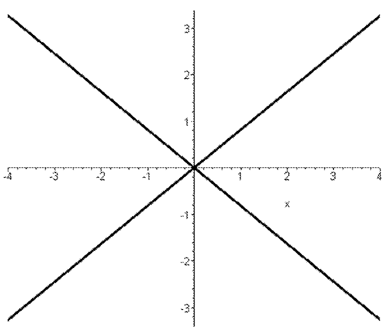
(b) 6 $-\frac{x^2}{9} + \frac{y^2}{4} = 1$

(c) 3 $r = 2\cos(4\theta)$

(d) 1 $\frac{x^2}{9} - \frac{y^2}{4} = 0$

(e) 2 $\frac{x^2}{9} + \frac{y^2}{4} = 1$

(f) 5 $x = -\frac{1}{4}y^2$



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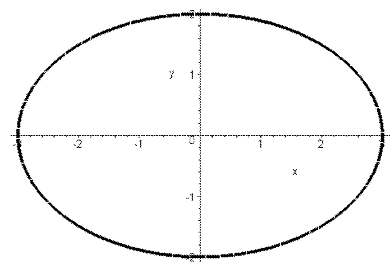

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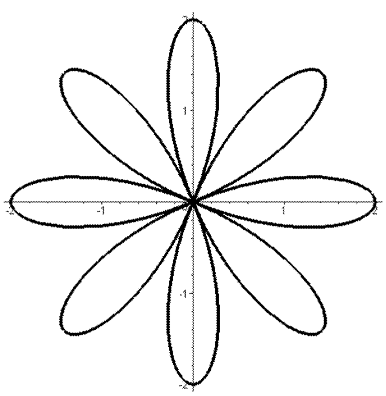

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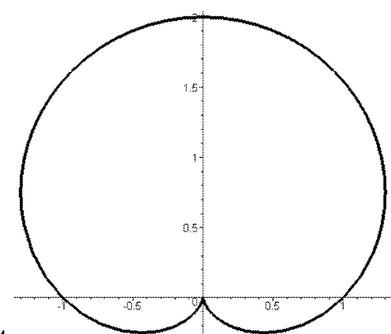

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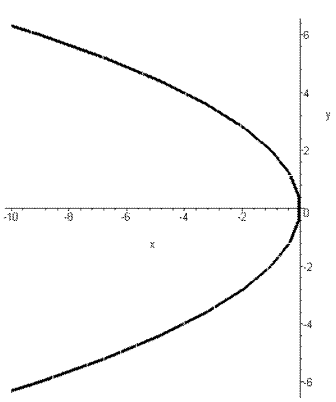

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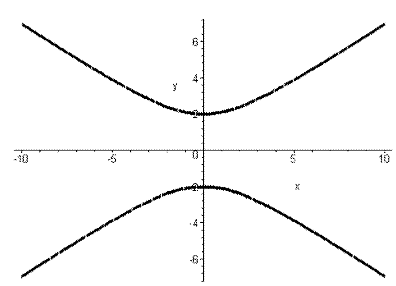

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Problem 2 (20 points)
Evaluate the integrals:

(a) $\int x^2 \cos(x)\, dx$

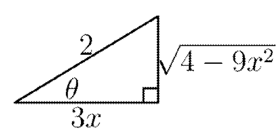
ANSWER: This integrand, a product, looks like a candidate for integration by parts. If we let $u = x^2$ and $dv = \cos(x)\, dx$ then $du = 2x\, dx$ and $v = \sin(x)$ so we have $\int x^2 \cos(x)\, dx = x^2 \sin(x) - 2\int x \sin(x)\, dx$. That new integral is not immediately solvable but it again looks like parts would help, and we have succeeded in reducing the degree from x^2 to x so maybe one more try will finish it off. This time we let $u = x$ and $dv = \sin(x)\, dx$ so $du = dx$ and $v = -\cos(x)$, and we have $\int x \sin(x)\, dx = -x\cos(x) + \int \cos(x)\, dx$. We can evaluate that integral directly, $\int \cos(x)\, dx = \sin(x) + C$, and putting the pieces back together we have $\int x^2 \cos(x)\, dx = x^2 \sin(x) - 2\left[-x\cos(x) + \sin(x) + C\right]$ $= x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + C$.

(b) $\int \tan^3(x)\, dx$

ANSWER: There are several ways to do this. I will use a trig substitution. Using the identity $\tan^2\theta + 1 = \sec^2\theta$ we have $\int \tan^3(x)\, dx = \int (\sec^2(x) - 1)\tan(x)\, dx$ That gives us two integrals to work out: $\int \tan(x)\sec^2(x)\, dx$, using the substitution $u = \tan x$ and $du = \sec^2(x)\, dx$ if you want to be formal, gives $\int \tan(x)\sec^2(x)\, dx = \frac{1}{2}\tan^2(x) + C$. $\int \tan(x)\, dx = \int \frac{\sin x}{\cos x} dx$, and the substitution $u = \cos x$ and $du = -\sin(x)\, dx$ gives $-\int \frac{1}{u} du = \ln|u| + C = -\ln|\cos x| + C$. Putting the pieces together we have $\int \tan^3(x)\, dx = \frac{1}{2}\tan^2(x) + \ln|\cos x| + C$.

Problem 3 (20 points)
Evaluate the integrals:

(a) $\int_0^{\frac{2}{3}} \sqrt{4 - 9x^2}\, dx$



ANSWER: This cries out for a trig substitution. Labelling a right triangle as shown at the right, $x = \frac{2}{3}\cos\theta$ and $\sqrt{4 - 9x^2} = 2\sin\theta$. Then $dx = -\frac{2}{3}\sin(\theta)d\theta$. Substituting we have $\int \sqrt{4 - 9x^2}\, dx = -\frac{4}{3}\int \sin^2(\theta)\, d\theta$, but we need to convert the limits on the integral. When $x = 0$ the triangle collapses to two vertical lines, with $\theta = \frac{\pi}{2}$. When $x = \frac{2}{3}$ the lower edge becomes as long as the hypoteneuse so $\theta = 0$. Hence we want to evaluate the integral $\int_0^{\frac{2}{3}} \sqrt{4 - 9x^2}\, dx = -\frac{4}{3}\int_{\frac{\pi}{2}}^{0} \sin^2(\theta)\, d\theta = \frac{4}{3}\int_0^{\frac{\pi}{2}} \sin^2(\theta)\, d\theta$. Next we use the identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, getting $\frac{2}{3}\int_0^{\frac{\pi}{2}} (1 - \cos 2\theta)\, d\theta = \frac{2}{3}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{2}{3}\left[(\frac{\pi}{2} - 0) - (0 - 0)\right] = \frac{\pi}{3}$.

(b) $\int_{-8}^{27} x^{-4}\, dx$

ANSWER: This is an improper integral: The integrand, which is $\frac{1}{x^4}$, "blows up" at $x = 0$ within the interval of integration. Hence we have to rewrite it as $\lim_{b\to 0^-}\int_{-8}^{b} x^{-4}dx + \lim_{a\to 0^+}\int_{a}^{27} x^{-4}dx$. Working on just the first one, using the power rule to integrate x^{-4}, we have $\lim_{b\to 0^-}\left[\frac{x^{-3}}{-3}\right]_{-8}^{b}$. Since the limit as $b \to 0^-$ of $\frac{1}{x^3}$ does not exist as a finite number, the integral does not converge. (Neither does the other integral, but the combination won't give a value if either diverges.) So this integral diverges.

Problem 4 (21 points)
For each of the following series, tell whether it converges or diverges. If it converges and it has some positive and some negative terms, tell whether it converges absolutely or conditionally. Be sure to give reasons justifying your answers.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2+1}$

ANSWER: These terms look, in size, a lot like $\frac{1}{n^2}$, so we try to compare to that series. But the comparison test requires positive terms, so we take the absolute values: $\left|\frac{(-1)^{n-1}}{n^2+1}\right| = \frac{1}{n^2+1} < \frac{1}{n^2}$, and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges as a p-series with $p = 2 > 1$. So the series we are asked about converges even when we take the absolute values, i.e. it converges absolutely. Since absolute convergence implies convergence, it also converges.

(b) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n-1}$

ANSWER: This is an alternating series, and the terms seem to be going to zero. More specifically, the size of each term, what the book calls a_n after separating off the sign as $(-1)^{somepower}$, is $\frac{1}{n-1}$: For n increasing toward ∞, that is continually decreasing with limit zero. Hence by the alternating series test (Leibniz' theorem) the series does converge. But if we take the absolute values of the terms, getting $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n-1}$, each term is greater than the corresponding term of the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ which we know diverges. So the series of absolute values diverges, hence the original series converges conditionally.

(c) $\displaystyle\sum_{n=1}^{\infty} n\,e^{-n^2}$

ANSWER: There are two fairly easy ways to do this one.

The exponential may suggest using calculus, and we can apply the integral test: This is a series of positive terms, and the function $f(x) = x\,e^{-x^2}$ fits the requirements of the integral test, so we try to evaluate the improper integral $\displaystyle\int_1^{\infty} x\,e^{-x^2}\,dx$. Using the substitution $u = -x^2$ we have $\displaystyle\lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^b$ which evaluates to $\frac{1}{2e}$, so the integral converges and hence the series does also.

Or, you can use the ratio test. The ratio of successive terms $\frac{a_{n+1}}{a_n}$ amounts to $\frac{n+1}{n}\frac{e^{-(n+1)^2}}{e^{-n^2}}$. The first of those fractions goes to 1 as $n \to \infty$. The second can be worked out as $\frac{e^{-n^2-2n-1}}{e^{-n^2}} = e^{-n^2-2n-1+n^2} = e^{-2n-1}$ and as $n \to \infty$ that goes to 0. Hence the number ρ for the ratio test is $0 < 1$ so the series converges.

Problem 5 (20 points)
Find the Taylor series at $a = 1$ for $f(x) = \cos(x^2 - 1)$. Show the terms through the 3^{rd} degree term.

Derive the coefficients from the general form for Taylor series, do not just "plug in" to some known series. That is, you should calculate a_0, a_1, a_2, and a_3 using derivatives, and show how they are fitted into a third-degree polynomial.

ANSWER: We will need the derivatives of $f(x)$ through the 3^{rd} derivative and the values of those derivatives at 1. Organized as a table we have

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$a_n = \frac{f^{(n)}(1)}{n!}$
0	$\cos(x^2-1)$	1	1
1	$-2x\sin(x^2-1)$	0	0
2	$-2\sin(x^2-1) - 4x^2\cos(x^2-1)$	-4	$\frac{-4}{2} = -2$
3	$-12x\cos(x^2-1) + 4x^2\sin(x^2-1)$	-12	$\frac{-12}{6} = -2$

Hence the initial terms of the series are

$$f(x) \approx 1 + 0(x-1) - 2(x-1)^2 - 2(x-1)^3.$$

Problem 6 (21 points)

Find all solutions of the differential equation

$$y'' + 2y' + y = 6\sin(2x).$$

ANSWER: This is a 2^{nd} order, constant coefficients, non-homogeneous, differential equation. First we find all solutions of the corresponding homogeneous equation $y'' + 2y' + y = 0$. The characteristic equation is $r^2 + 2r + 1 = 0$. You could use the quadratic formula to find the roots of that polynomial, but it factors nicely as $(r+1)^2$ so the roots are -1 and -1, i.e. -1 is a root of multiplicity 2. We can write the general solution to the homogeneous equation as $y_h = C_1e^{-x} + C_2\,x\,e^{-x}$.

Now we need to find some solution y_p to the non-homogeneous equation. We assume it can be found as $y_p = A\cos(2x) + B\sin(2x)$ for some numbers A and B. Then $y_p' = -2A\sin(2x) + 2B\cos(2x)$ and $y_p'' = -4A\cos(2x) - 4B\sin(2x)$. The fact that y_p fits the equation requires $-4A\cos(2x) - 4B\sin(2x) + 2(-2A\sin(2x) + 2B\cos(2x)) + A\cos(2x) + B\sin(2x) = 6\sin(2x)$. That in turn requires that the $\sin(2x)$ terms from both sides agree and likewise the $\cos(2x)$ terms: From the $\cos(2x)$ terms we get $-4A + 4B + A = 0$, and from the $\sin(2x)$ terms we have $-4B - 4A + B = 6$. Simplifying, we get $-3A + 4B = 0$ and $-4A - 3B = 6$. You can use your favorite method of solving two linear equations in two unknowns to find A and B: One way is to multiply the first equation by 4 and the second by -3 and add the results, so that the A terms cancel out. The results are $A = -\frac{24}{25}$ and $B = -\frac{18}{25}$, so $y_p = -\frac{24}{25}\cos(2x) - \frac{18}{25}\sin(2x)$.

We combine these to get the general solution as $y_h + y_p = C_1e^{-x} + C_2\,x\,e^{-x} - \frac{24}{25}\cos(2x) - \frac{18}{25}\sin(2x)$.

Problem 7 (20 points)

Let $\vec{u} = 2\vec{\imath} + \vec{\jmath} + \vec{k}$ and $\vec{v} = \vec{\imath} - 2\vec{\jmath} + 2\vec{k}$.

(a) What is $|\vec{v}|$, the magnitude of $\vec{v}$?

ANSWER: $|\vec{v}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$.

(b) What is the $\cos\theta$, if θ is the angle between $\vec{u}$ and $\vec{v}$?

ANSWER: $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{2\times1+1\times(-2)+1\times2}{\sqrt{2^2+1^2+1^2}\times3} = \frac{2}{3\sqrt{6}}$. You can rewrite that as $\frac{\sqrt{2}}{3\sqrt{3}}$ or $\frac{\sqrt{2}\sqrt{3}}{9}$ or yet other forms.

(c) What is the scalar component of $\vec{u}$ in the direction of $\vec{v}$?

ANSWER: The scalar component will be $(\cos\theta)|\vec{u}| = \frac{2}{3}$.

(d) What is $\text{proj}_{\vec{v}}\vec{u}$, the projection of $\vec{u}$ on $\vec{v}$?

ANSWER: The vector projection will have magnitude the scalar component and direction determined by a unit vector in the direction of $\vec{v}$. That can be written as $\frac{2}{3} \times \frac{1}{|\vec{v}|}\vec{v} = \frac{2}{9}\vec{v} = \frac{2}{9}\vec{\imath} - \frac{4}{9}\vec{\jmath} + \frac{4}{9}\vec{k}$.

(e) Find two vectors $\vec{u}_1$ and $\vec{u}_2$ such that (i) $\vec{u} = \vec{u}_1 + \vec{u}_2$, (ii) $\vec{u}_1$ is parallel to $\vec{v}$, and (iii) $\vec{u}_2$ is orthogonal to $\vec{v}$.

ANSWER: We can use for $\vec{u}_1$ the vector $\text{proj}_{\vec{v}}\vec{u}$ that we just computed, $\vec{u}_1 = \frac{2}{9}\vec{v} = \frac{2}{9}\vec{\imath} - \frac{4}{9}\vec{\jmath} + \frac{4}{9}\vec{k}$. Then since $\vec{u} = \vec{u}_1 + \vec{u}_2$, we must have $\vec{u}_2 = \vec{u} - \vec{u}_1 = (2\vec{\imath} + \vec{\jmath} + \vec{k}) - (\frac{2}{9}\vec{\imath} - \frac{4}{9}\vec{\jmath} + \frac{4}{9}\vec{k}) = \frac{16}{9}\vec{\imath} + \frac{13}{9}\vec{\jmath} + \frac{5}{9}\vec{k}$.
You can check that $\vec{u}_1 \cdot \vec{u}_2 = 0$, so they are orthogonal, and that their sum really is $\vec{u}$.

Problem 8 (21 points)
Suppose the polynomial $1 - \frac{x^2}{2} + \frac{x^4}{24}$ is used to calculate, approximately, $\cos(x)$. If this will be used for values of x from -1 to 1, what accuracy can you guarantee will be achieved?
Your answer should use the remainder term from Taylor's theorem in showing your answer is valid. If you know another mathematically correct way to do the problem you can use that as a check on your answer and get up to 5 extra points. But it will not substitute for an answer using Taylor's theorem.
ANSWER: This polynomial is the first few terms of the Maclaurin series for $\cos(x)$. For any number x the even powers will be positive and the alternating signs in the series will make this an alternating series, so one approach would be to use the alternating series test and estimate of error. But we are told explicitly to use the remainder term from Taylor's theorem, $R_n = \frac{f^{(n+1)}(c)x^{n+1}}{(n+1)!}$ where c is some number between 0 and x. The values of x for which we need to do this are between -1 and 1 so all we know about c is that it also is between -1 and 1.
What n should we use? The polynomial presents the terms of the Maclaurin series through the 4^{th} degree term, but that is the same as the terms through the 5^{th} degree term since the 5^{th} degree term in this series is $0x^5$. So we can use $n = 5$ and achieve a better estimate than if we used $n = 4$. We need the $(n+1)^{st} = 6^{th}$ derivative of $f(x) = \cos(x)$, to be evaluated at c. The 6^{th} derivative is actually $-\cos(x)$ but any derivative is $\pm\cos(x)$ or $\pm\sin(x)$ so about the best we can say about the derivative evaluated at c is that it is between ± 1, the range of the sine and cosine functions. So we get that $|R_5| \leq \frac{1 \times |x|^6}{6!}$. How big can $|x|^6$ be? Again we use the fact that $x \in [-1, 1]$ to say $|x|^6 \leq 1$. So the error is at most $|R_5| \leq \frac{1}{6!} = \frac{1}{720} \approx 0.00139$.
(If we had used the alternating series approach, we would say the error was at most the absolute value of the first omitted term. That would be at most $|-\frac{x^6}{6!}| \leq \frac{1}{720}$ so we get the same answer. From this approach we would get one more bit of information: Since that first omitted term is negative, for any x, the polynomial will give an approximation that errs by being too large rather than too small.)

Problem 9 (21 points)
Solve the initial value problem

$$x\frac{dy}{dx} + 2y = x^3 \qquad \text{(for } x > 0\text{)} \qquad \text{and} \qquad y(2) = 1.$$

ANSWER: The equation is a first order linear equation. If we rewrite it as $\frac{dy}{dx} + \frac{2}{x}y = x^2$, dividing by x which is legal since we are told $x > 0$, we see it in our standard form $y' + P(x)y = Q(x)$ where $P(x) = \frac{2}{x}$ and $Q(x) = x^2$. If you now just apply our routine procedure you first evaluate $\int P(x)\,dx = 2\int \frac{dx}{x} = 2\ln|x|$ throwing away the constant. We can drop the absolute values since $x > 0$. Now we raise e to that power to get the integrating factor $\nu(x) = e^{2\ln x} = (e^{\ln x})^2 = x^2$. You can either multiply that onto the original equation to get the left side to be the derivative of a product or continue in formula-using mode to write the general solution as $y(x) = \frac{1}{\nu(x)}\int \nu(x)Q(x)\,dx = \frac{1}{x^2}\int x^2 \times x^2\,dx = x^{-2}\int x^4\,dx = x^{-2}(\frac{x^5}{5} + C) = \frac{x^3}{5} + Cx^{-2}$.

Now we have to choose the value of C that makes this function fit the initial conditions. From $y(2) = 1$ we have $1 = \frac{2^3}{5} + \frac{C}{4}$, or $C = 4 \times (1 - \frac{8}{5}) = -\frac{12}{5}$. Putting that in the solution we have $y(x) = \frac{x^3}{5} - \frac{12}{5x^2}$.

Problem 10 (21 points)
Consider two planes, Π_1 and Π_2, given by

$$\Pi_1: \qquad x + 2y - z = 7$$

and

$$\Pi_1: \qquad 2x + 3y + 2z = 4.$$

(a) Find parametric equations for the line of intersection of these two planes.
Hint: The point $(1, 2, -2)$ is on both planes.

ANSWER: To find equations for the line we want a point on it and a vector in its direction: We are given that $(1, 2, -2)$ works for the point. Since the line lies in each plane, it is perpendicular to the normal to either plane. The vector $\vec{n}_1 = \vec{\imath} + 2\vec{\jmath} - \vec{k}$ is normal to the plane Π_1 and the vector $\vec{n}_2 = 2\vec{\imath} + 3\vec{\jmath} + 2\vec{k}$ is normal to Π_2. If we let $\vec{v} = \vec{n}_1 \times \vec{n}_2$ it will be perpendicular to both normals and hence in the right direction for the line. Computing, $\vec{n}_1 \times \vec{n}_2 = (\vec{\imath} + 2\vec{\jmath} - \vec{k}) \times (2\vec{\imath} + 3\vec{\jmath} + 2\vec{k}) = 7\vec{\imath} - 4\vec{\jmath} - \vec{k}$. (You can check that this is perpendicular to each of those normal vectors by computing dot products.) Now that we know a direction vector we can write parametric equations for the line as
$x = 7t + 1$
$y = -4t + 2$
$z = -t - 2$
where $-\infty < t < \infty$.

(b) The angle between two planes means the angle between a vector perpendicular to one plane and a vector perpendicular to the other. What is the cosine of the angle between Π_1 and Π_2?

ANSWER: Let θ be the angle between the planes. We could use either the dot product or the cross product to find θ. Usually we use the dot product since the cross product is harder to compute, but in this case we have already computed the cross product $\vec{n}_1 \times \vec{n}_2$ so we could really do it either way. But the cross product would give us $\sin\theta$ and we would have to get from that to the cosine using a trig identity, so I'll go ahead with the dot product: $\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \dfrac{1 \times 2 + 2 \times 3 + (-1) \times 2}{\sqrt{1+4+1}\sqrt{4+9+4}} = \dfrac{6}{\sqrt{6}\sqrt{17}} = \dfrac{\sqrt{6}}{\sqrt{17}} \approx 0.594.$