Your Name: ________________________________

Circle your TA's name:

Lino Amorim	**Jon Godshall**	**Ed Hanson**
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Mathematics 222, Spring 2007 **Lecture 3 (Wilson)**

Second Midterm Exam April 12, 2007

There is a problem on the back of this sheet! Do not accidentally skip over it!

Write your answers to the eight problems in the spaces provided. If you must continue an answer somewhere other than immediately after the problem statement, be sure (a) to tell where to look for the answer, and (b) to label the answer wherever it winds up. In any case, be sure to make clear what is your final answer to each problem.

Wherever possible, leave your answers in exact forms (using $\frac{\pi}{3}$, $\sqrt{3}$, $\cos(0.6)$, and similar numbers) rather than using decimal approximations. For example, $\sin(\frac{\pi}{6}) = \frac{1}{2}$, and writing something like $.499$ may not get you full credit. If you use a calculator to evaluate your answer be sure to show what you were evaluating!

There are some formulae at the end of the exam that you may wish to use.

You may refer to notes you have brought on one or two sheets of paper, as announced in class and by email.

BE SURE TO SHOW YOUR WORK, AND EXPLAIN WHAT YOU DID. YOU MAY RECEIVE REDUCED OR ZERO CREDIT FOR UNSUBSTANTIATED ANSWERS. ("I did it on my calculator" and "I used a formula from the book" (without more details) are not sufficient substantiation...)

Problem	Points	Score
1	13	
2	12	
3	12	
4	12	
5	12	
6	13	
7	13	
8	13	
TOTAL	100	

Problem 1 (13 points)
Find the area of the region which is inside the circle $r = -2\cos\theta$ but outside the circle $r = 1$.

Problem 2 (12 points)

(a) Let $a_n = \frac{\ln(n+1)}{\sqrt{n}}$. Does the sequence $\{a_n\}$ converge or diverge? If it converges, what is its limit?

(b) The series $\displaystyle\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$ is a geometric series, for any given x. For what values of x does this series converge? For those values of x that do make it converge, what does it converge to?

(c) Let $a_1 = 2$ and for $n \geq 1$ let $a_{n+1} = \frac{1+\sin n}{n} a_n$. Does the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converge or diverge? Be sure to give a reason!

Problem 3 (12 points)

(a) Does the series $\sum_{n=1}^{\infty} \frac{1-n}{n\,2^n}$ converge or diverge? Be sure to give reasons!

(b) Does the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converge absolutely, conditionally, or not at all? Be sure to give reasons!

Problem 4 (12 points)

We know that the so-called p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p < 1$. Use the integral test to show that this is true.

You may ignore the cases $p \leq 0$ and $p = 1$.

(Be sure to explain your steps and to show why the the integral test does apply. You may assume that x^p increases as x increases, which is true for $x \geq 1$ and $p > 0$, but you should refer to that assumption wherever it may be useful.)

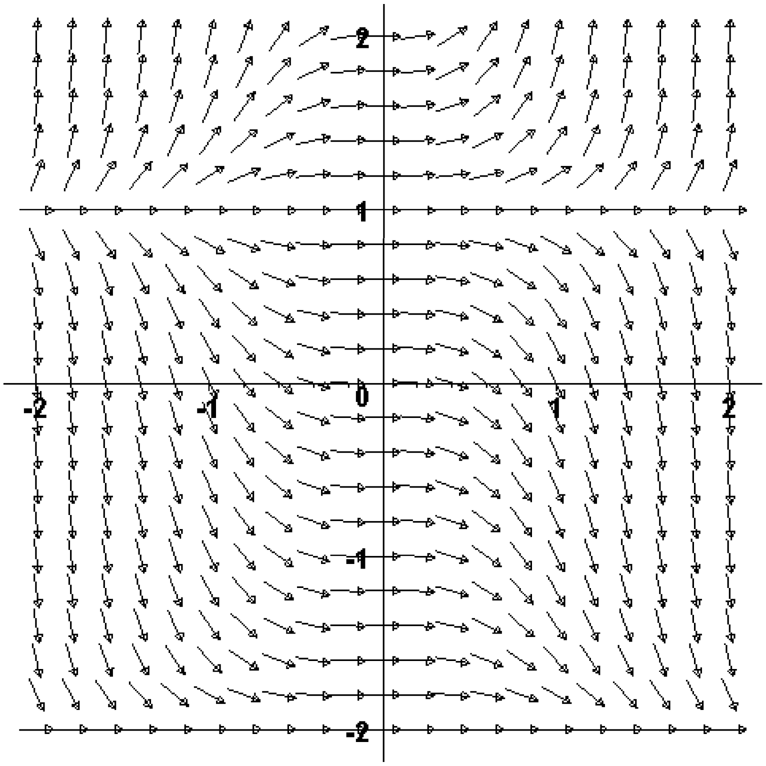
Problem 5 (12 points)
Find the Maclaurin series (the Taylor series with $a = 0$) for $f(x) = \sin 3x$.
(Do explicitly derive the coefficients: Do not just substitute into the known series for $\sin x$. Write out the terms through the 7^{th} power of x, and show (without proof) what the general term looks like.)

Problem 6 (13 points)

(a) At the right is a portion of the slope field for the equation

$$\frac{dy}{dx} = x^2(y-1)(y+2).$$

On that field draw two graphs, one the solution corresponding to $y(-1) = \frac{1}{2}$ and the other the solution corresponding to $y(0) = 1$.



(b) Solve the differential equation $\sqrt{x}\,\frac{dy}{dx} = e^y\, e^{\sqrt{x}}$, with $x > 0$.

(For full credit, find explicitly functions $y(x)$ which solve the equation. A relation involving y and x but no derivatives can get partial credit.)

Problem 7 (13 points)
Suppose we use the approximation $e^x = 1 + x + \frac{x^2}{2}$ (the first three terms of the Maclaurin series for e^x) when x is small: Use the remainder term from Taylor's theorem to find a bound (i.e. a maximum possible value) for the error in this approximation if we restrict its use to $|x| < 0.1$. (You can use the fact that $e < 3$ if that is helpful. Your answer should include both a number (some fraction, perhaps) such that the error can be guaranteed not to exceed that number as well as your argument showing why the error really does not exceed that number. Do not assume you know exactly $e^{0.1}$ or any power of e other than $e^0 = 1$.)

Problem 8 (13 points)

Solve the initial value problem

$$(x+1)\frac{dy}{dx} - 2x(x+1)y = \frac{e^{x^2}}{x+1} \qquad \text{(for } x > -1\text{)} \qquad \text{with } y(0) = 5.$$

Some formulas, identities, and numeric values you might find useful:

Values of trig functions:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	—

Trig facts:

1. $\tan\theta = \frac{\sin\theta}{\cos\theta}$
2. $\sec\theta = \frac{1}{\cos\theta}$
3. $\sin^2\theta + \cos^2\theta = 1$
4. $\sec^2\theta = \tan^2\theta + 1$
5. $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
6. $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
7. $\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$
8. $\sin^2 x = \frac{1}{2}(1-\cos 2x)$
9. $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

Derivative formulas:

1. $\frac{d}{dx}\tan x = \sec^2 x$
2. $\frac{d}{dx}\sec x = \sec x \tan x$
3. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
4. $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
5. $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$
6. $\frac{d}{dx}\ln x = \frac{1}{x}$
7. $\frac{d}{dx}e^x = e^x$

Integral formulas:

1. $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C$, if $n \neq -1$
2. $\int \frac{1}{u}\,du = \ln|u| + C$
3. $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$
4. $\int \frac{du}{1+u^2} = \tan^{-1} u + C$
5. $\int \sec(u)\,du = \ln|\sec(u)+\tan(u)| + C$
6. $\int u\,dv = uv - \int v\,du$

Algebra formulas:

1. $\ln(xy) = \ln(x) + \ln(y)$
2. $a^{x+y} = a^x a^y$
3. $a^b = e^{b\ln a}$