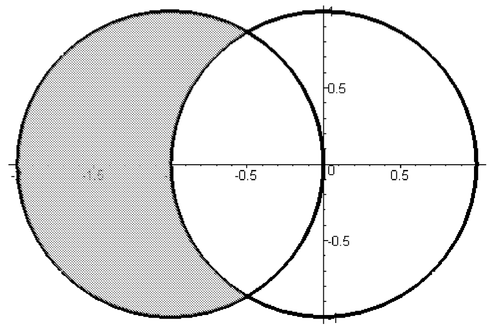
Mathematics 222, Spring 2007 | Lecture 3 (Wilson)

Second Midterm Exam | April 12, 2007 | ANSWERS

Problem 1 (13 points)
Find the area of the region which is inside the circle $r = -2\cos\theta$ but outside the circle $r = 1$.

ANSWER: The region is shaded in the picture to the right. You did not have to draw a picture, but some sketch helps in determining where the two circles intersect. We see that there is one intersection point above and one below the x-axis, that they appear to be on a vertical line, and that they appear to be more than 45° above or below the x-axis. Solving $-2\cos\theta = 1$, $\cos\theta = -\frac{1}{2}$, we find $\theta = \pm\frac{2\pi}{3}$, which fits those observations. If we re-describe the lower point as $\theta = \frac{4\pi}{3}$ we see that the region corresponds to $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$.



For any θ value in that interval, the region has its inner radius $r_1 = 1$ and its outer radius $r_2 = -2\cos\theta$. Hence we can find the area by evaluating the integral $\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (r_2^2 - r_1^2)\, d\theta = \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \left(4\cos^2\theta - 1\right) d\theta$. Using the identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, we get $\frac{\sqrt{3}}{2} + \frac{\pi}{3}$.

Problem 2 (12 points)

(a) Let $a_n = \frac{\ln(n+1)}{\sqrt{n}}$. Does the sequence $\{a_n\}$ converge or diverge? If it converges, what is its limit?

ANSWER:
As $n \to \infty$, both the numerator and denominator also go to ∞. So this comes down to which goes faster.... We can think of this in terms of continuous functions, $\lim_{x\to\infty} \dfrac{\ln(x+1)}{\sqrt{x}}$, where x is not restricted to whole number values, which will have the same answer. Then we see the form $\frac{\infty}{\infty}$ which suggests using l'Hopital's rule. Taking derivatives (separately, not as a fraction) of the numerator and denominator we have $\lim_{x\to\infty} \dfrac{1/(x+1)}{1/(2\sqrt{x})} = \lim_{x\to\infty} \dfrac{2\sqrt{x}}{x+1}$ $< \lim_{x\to\infty} \dfrac{2\sqrt{x+1}}{x+1} = \lim_{x\to\infty} \dfrac{2}{\sqrt{x+1}}$. At this point I think it is fair to say that the limit is clearly 0: The numerator is not changing, while the denominator is growing without bound. So the answer is 0.

(b) The series $\displaystyle\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$ is a geometric series, for any given x. For what values of x does this series converge? For those values of x that do make it converge, what does it converge to?

ANSWER:
The ratio r that is multiplied onto any term to get the next one is $\frac{x-1}{2}$. We know that a geometric series converges only when $|r| < 1$, i.e. $-1 < \frac{x-1}{2} < 1$, $-2 < x - 1 < 2$,

$-1 < x < 3$. We could also write that as an interval, the series converges if and only if $x \in (-1, 3)$.

The first term of the series, when $n = 0$, is 3, which is the a we referred to in talking about geometric series. When the series does converge, it converges to $\frac{a}{1-r} = \frac{3}{1-(x-1)/2} = \frac{6}{3-x}$.

(c) Let $a_1 = 2$ and for $n \geq 1$ let $a_{n+1} = \frac{1+\sin n}{n} a_n$. Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Be sure to give a reason!

ANSWER:
We test for convergence of the absolute values using the ratio test. (That way we don't have to worry about whether the series has positive terms, which the ratio test required. You could also note that $\sin n < 1$ for any positive whole number n and confirm that the terms are actually positive.) Then the ratio $\frac{|a_{n+1}|}{|a_n|} = \left|\frac{1+\sin n}{n}\right| < \frac{2}{n}$ which clearly goes to 0 as $n \to \infty$. So the limiting ratio ρ is 0 and the series converges.

Problem 3 (12 points)

(a) Does the series $\sum_{n=1}^{\infty} \frac{1-n}{n\,2^n}$ converge or diverge? Be sure to give reasons!

ANSWER:
There are several ways to work on this. The first thing that occurs to me is to take the fraction apart: $\sum_{n=1}^{\infty} \frac{1-n}{n\,2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{n\,2^n} - \frac{1}{2^n} \right) =$ (if they converge) $\sum_{n=1}^{\infty} \frac{1}{n\,2^n} - \sum_{n=1}^{\infty} \frac{1}{2^n}$ Now the second of those series is geometric with ratio $r = \frac{1}{2} < 1$, so it converges. In the first series, $\frac{1}{n\,2^n} < \frac{1}{2^n}$, and we noted that the geometric series with ration $r = \frac{1}{2}$ converges, so the first series converges by the comparison test. Hence the original series can be written as the difference of these two converging series and it will converge.

(b) Does the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converge absolutely, conditionally, or not at all? Be sure to give reasons!

ANSWER:
As $n \to \infty$, $\frac{1}{\sqrt{n}} \to 0$, decreasing consistently, so by the Alternating Series Test (Leibniz' theorem) this series does converge as given. But if we take the absolute values we get $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a p-series with $p = \frac{1}{2} < 1$ and so diverges. Hence the series converges but not absolutely, so it converges conditionally.

Problem 4 (12 points)

We know that the so-called p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p < 1$. Use the integral test to show that this is true.
You may ignore the cases $p \leq 0$ and $p = 1$.

(Be sure to explain your steps and to show why the the integral test does apply. You may assume that x^p increases as x increases, which is true for $x \geq 1$ and $p > 0$, but you should refer to that assumption wherever it may be useful.)

ANSWER:
The integral test applies to series of positive terms: Clearly n^p and hence $\frac{1}{n^p}$ are positive numbers for all positive numbers n, for any number p.
Let $f(x) = \frac{1}{x^p}$ for $x \geq 1$. Since we are allowed to assume x^p is increasing, without bound, $f(x)$ is a decreasing function for $x \geq 1$, and $\lim_{x\to\infty} f(x) = 0$. It is also continuous and positive, and for our series $a_n = f(n)$ for all whole numbers $n \geq 1$. So by the integral test the series will converge if and only if $\int_1^\infty f(x)\,dx$ converges. For $p \neq 1$: $\int_1^\infty f(x)\,dx = \int_1^\infty x^{-p}\,dx$ $= \lim_{b\to\infty} \int_1^b x^{-p}\,dx = \lim_{b\to\infty} \left[\frac{x^{(1-p)}}{1-p}\right]_1^b = \lim_{b\to\infty} \frac{b^{(1-p)}}{1-p} - \frac{1}{1-p}$. If $p < 1$, $1-p > 0$ and $b^{(1-p)} \to \infty$, so the integral diverges. If $p > 1$, $1-p < 0$ and $b^{(1-p)} \to 0$, so the integral converges.
You were told you did not have to consider the $p = 1$ case, but for completeness I will include it: We can either go back and redo the integral for this case, getting a logarithm from the integration and taking a limit, or say that we already know that the harmonic series diverges (which was shown two ways, one of which used the integral test but not anything about p-series so this is not a circular argument).
So now we have checked all the required cases: The series diverges if $0 < p \leq 1$ and converges if $p > 1$.

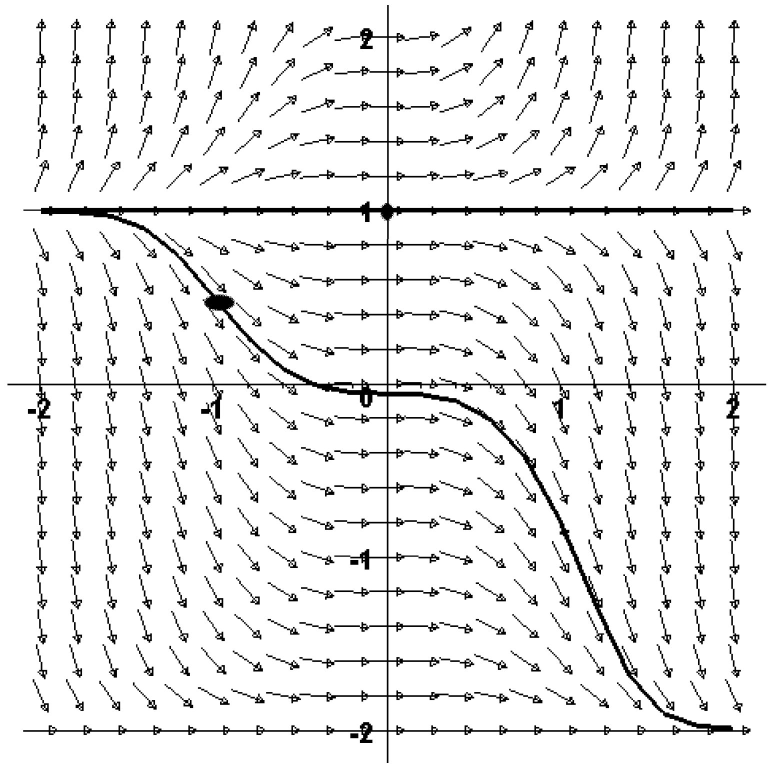
Problem 5 (12 points)
Find the Maclaurin series (the Taylor series with $a = 0$) for $f(x) = \sin 3x$.
(Do explicitly derive the coefficients: Do not just substitute into the known series for $\sin x$. Write out the terms through the 7^{th} power of x, and show (without proof) what the general term looks like.)

ANSWER:
The first several derivatives (starting with the 0^{th} derivative, i.e. just the function) of $f(x)$ are $\sin 3x$, $3\cos 3x$, $-9\sin 3x$, $-27\cos 3x$, $81\sin 3x$... It is apparent that the n^{th} derivative will involve $\sin 3x$ for any even n and $\cos 3x$ for odd n, and that the signs will alternate in steps of two positive, then two negative, etc. The derivative will be multiplied by 3^n since the chain rule brings out one more 3 each time we differentiate.
In computing the coefficients a_n for the series, we note that those derivatives will be evaluated at 0 since this is a Maclaurin series. Then the terms with $\sin 3x$, the even power terms, will have coefficeient $a_n = 0$, since $\sin(3 \times 0) = 0$. The odd power terms will have $a_n = \pm\frac{3^n \times 1}{n!}$ since $\cos(3 \times 0) = 1$. The sign will be positive on the terms with x^1, x^5, x^9, and so on where the powers are increasing by 4, and negative on the ones in between. So we the first terms are $3^1 x - 3\frac{3x^3}{3!} + \frac{3^5 x^5}{5!} - \frac{3^7 x^7}{7!}$ or $3x - \frac{27}{6}x^3 + \frac{243}{120}x^5 - \frac{2187}{5040}x^7$. The general term is $a_n = \pm\frac{3^n}{n!}x^n$ for odd n, where the sign is positive for $n = 1, 5, 9, 13, \ldots$ and negative for $n = 3, 7, 11, 15, \ldots$, and $a_n = 0$ for even n.

Problem 6 (13 points)

(a) On that field draw two graphs, one the solution corresponding to $y(-1) = \frac{1}{2}$ and the other the solution corresponding to $y(0) = 1$.

ANSWER: The dots on the field show the initial conditions, and the curves going through them show the corresponding solutions.



ANSWER:

(b) Solve the differential equation $\sqrt{x}\,\frac{dy}{dx} = e^y\, e^{\sqrt{x}}$, with $x > 0$.

(For full credit, find explicitly functions $y(x)$ which solve the equation. A relation involving y and x but no derivatives can get partial credit.) ANSWER:
Since $x > 0$ we can divide by $\sqrt{x}$, and multiply by e^{-y}, getting $e^{-y}dy = \frac{1}{\sqrt{x}}e^{\sqrt{x}}dx$. Integrating both sides (use $u = \sqrt{x}$ on the right side) we get $-e^{-y} = 2e^{\sqrt{x}} + C$. Thus $e^{-y} = -2e^{\sqrt{x}} + C$, $-y = \ln\left(-2e^{\sqrt{x}} + C\right)$, $y = -\ln\left(-2e^{\sqrt{x}} + C\right)$, and so

$$y = \ln \frac{1}{-2e^{\sqrt{x}} + C}.$$

Problem 7 (13 points)
Suppose we use the approximation $e^x = 1 + x + \frac{x^2}{2}$ (the first three terms of the Maclaurin series for e^x) when x is small: Use the remainder term from Taylor's theorem to find a bound (i.e. a maximum possible value) for the error in this approximation if we restrict its use to $|x| < 0.1$. (You can use the fact that $e < 3$ if that is helpful. Your answer should include both a number (some fraction, perhaps) such that the error can be guaranteed not to exceed that number as well as your argument showing why the error really does not exceed that number. Do not assume you know exactly $e^{0.1}$ or any power of e other than $e^0 = 1$.)

ANSWER:
We use the remainder term $R(x, n)$ for $n = 2$, with $-0.1 < x < 0.1$. The formula for the remainder term is $\frac{f^{(n+1)}(c)x^{n+1}}{(n+1)!}$ which for $n = 2$ and $f(x) = e^x$ works out to $\frac{e^c x^3}{3!}$, where c is some

number between 0 and x. Calling that remainder R, we need to find the largest $|R|$ can be for $-0.1 < x < 0.1$. Since e^x is an increasing function, the largest e^c can be happens for c near 0.1, i.e. $e^c < e^{0.1}$. x^3 is also an increasing function, so $x^3 < (0.1)^3$ when $-0.1 < x < 0.1$. Hence $R < \frac{e^{0.1}(0.1)^3}{6}$. But we don't know $e^{0.1}$ so we note that it would be at worst $3^{0.1}$ and claim as our answer that $|R| < \frac{(3)^{0.1}\times(0.001)}{6}$. That is a fine version of an answer. If you evaluate it on a calculator you get $0.000186\ldots$.

Problem 8 (13 points)
Solve the initial value problem

$$(x+1)\frac{dy}{dx} - 2x(x+1)y = \frac{e^{x^2}}{x+1} \qquad \text{(for } x > -1\text{)} \qquad \text{with } y(0) = 5.$$

ANSWER:
First we get the general solution to the differential equation, then we will choose a constant value that satisfies the initial condition. If we divide by $x+1$ (legitimate since $x > -1$) we have $\frac{dy}{dx} - 2xy = \frac{e^{x^2}}{(x+1)^2}$, a first order linear differential equation in the form we have treated. The part we have called $P(x)$ is $-2x$, and $Q(x) = \frac{e^{x^2}}{(x+1)^2}$. Computing $\int -2x\,dx$ we get $-x^2 + C$, and we drop the $+C$. Then the integrating factor that the book calls $\nu(x)$ is e^{-x^2}. We can then either multiply the whole equation by $\nu(x)$ and recognize the left side as the derivative of a product, then integrate, or "plug" into the formula $y = \frac{1}{\nu(x)}\int \nu(x)Q(x)\,dx$ which gets us to the same thing. Then $y = e^{x^2}\int \frac{dx}{(x+1)^2} = e^{x^2}\left(-\frac{1}{x+1} + C\right)$.
Now using the initial condition $y(0) = 5$, $5 = 1 \times (-1 + C)$ so $C = 6$, and the solution is $y = e^{x^2}\left(-\frac{1}{x+1} + 6\right)$.