Mathematics 222, Spring 2007 Lecture 3 (Wilson)

First Midterm Exam March 1, 2007 ANSWERS

Notice that some problems can be done in various ways, and I have given only one possible answer. Also remember that answers could be correct despite looking quite different from those given: In particular this applies to indefinite integrals, where the "$+C$" can take many forms. For example, $\sin^{-1}\theta = -\cos^{-1}\theta +$ a constant that could be included in $+C$.

Problem 1

Evaluate the integrals:

(a) $\displaystyle\int_1^2 x\,\ln(x)\,dx$

ANSWER:

We can do this using integration by parts. If we let $u = \ln x$, then $du = \frac{1}{x}\,dx$. Now dv has to be whatever is left to make up the integrand, so $dv = x\,dx$, hence $v = \frac{x^2}{2}$. Using the integration by parts formula $\int u\,dv = uv - \int v\,du$, we have $\displaystyle\int_1^2 x\,\ln(x)\,dx = \left[(\ln x)(\frac{x^2}{2})\right]_1^2 - \int_1^2 (\frac{x^2}{2})(\frac{1}{x})\,dx$ $= \left[(\ln x)(\frac{x^2}{2})\right]_1^2 - \left[\frac{x^2}{4}\right]_1^2 = (2\ln 2 - \frac{1}{2}\ln 1) - (2 - \frac{1}{4}) = 2\ln 2 - \frac{3}{4}$.

(b) $\displaystyle\int \cos^3(x)\,dx$

ANSWER:

This odd power of the cosine can be integrated by keeping one copy of cosine to go with dx and replacing $\cos^2 x$ by $1-\sin^2 x$, then (if you want to be formal) substituting $u = \sin x$ so $du = \cos(x)\,dx$. We have $\displaystyle\int \cos^3(x)\,dx = \int (1-\sin^2 x)\cos(x)\,dx = \int \cos(x)\,dx - \int \sin^2(x)\cos(x)\,dx =$ $\sin(x) - \dfrac{1}{3}\sin^3(x) + C$. (Note that this answer can be written in other ways that don't look at all the same, using trig identities to produce an answer that differs by a constant which is swallowed in C.)

Problem 2

Evaluate the integral $\displaystyle\int \frac{5x-3}{(x+1)(x-3)}\,dx$.

ANSWER:

This seems to call for a partial fraction rewriting of the integrand. We know we can write $\frac{5x-3}{(x+1)(x-3)}\,dx = \frac{A}{x+1} + \frac{B}{x-3}$, for some constants A and B. Multiplying both sides by $(x+1)(x-3)$ we have $5x-3 = A(x-3) + B(x+1)$. Expanding we get $5x-3 = (A+B)x + (-3A+B)$. Equating the x terms we have $A+B=5$, and from the constants we have $-3A+B=-3$. You can solve these for A and B in several ways. One way: Multiply $A+B=5$ by 3 to get $3A+3B=15$. Add that equation to $-3A+B=-3$ and you get $4B=12$, so $B=3$. Putting that in $A+B=5$ gives $A=2$.

So now we know $\displaystyle\int \frac{5x-3}{(x+1)(x-3)}\,dx = \int \frac{2}{x+1}dx + \int \frac{3}{x-3}dx$. Each of those integrals is of the form $\int \frac{1}{u}\,du$, so the answer is $2\ln|x+1| + 3\ln|x-3| + C$.

Problem 3

A parabola centered at $(0, 0)$ and opening upwards goes through the point $(-4, 1)$.
Find an equation for this curve. What are the coordinates of its focus? (Write out the equation and the coordinates explicitly!)

ANSWER:

We know we can write the equation for a parabola that is in standard position and opens upward as $y = \frac{1}{4p}x^2$, for some number p. But since $(-4, 1)$ is on the curve, $1 = \frac{1}{4p}(-4)^2 = \frac{4}{p}$. Hence $p = 4$, and the equation for the parabola is $y = \frac{1}{16}x^2$.
We also know that a parabola in this position has its focus at $(0, p)$ on the y-axis, i.e. at $(0, 4)$.

Problem 4

A function $f(x)$ obtained from real-world measurements takes on these values:

x	$f(x)$
1	3
2	2
3	3
4	1

Estimate $\int_1^4 f(x)\,dx$ using one of our numerical integration techniques, Simpson's Rule or the trapezoidal Rule: Be sure to specify which you are using!

ANSWER:

We have the interval $[1, 4]$ divided into $n = 3$ subintervals. Simpson's rule only works for an even number of subintervals so we have to use the trapzoidal rule. (Since most people think that is easier to use, I suspect most people taking the test would have gone this way even without that argument!) Our subinterval end points are $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, and $x_3 = 4$. The corresponding function values are $f(1) = 3$, $f(2) = 2$, $f(3) = 3$, and $f(4) = 1$. The length of each subinterval is $\Delta x = 1$. Using the trapezoidal rule we want to calculate $\frac{\Delta x}{2}(f(1) + 2f(2) + 2f(3) + f(4)) = \frac{1}{2}(3+4+6+1) = \frac{1}{2} \times 14 = 7$.

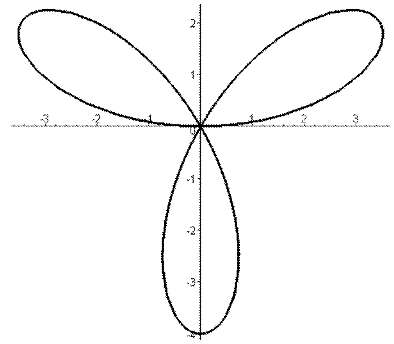
Problem 5

Find parametric equations $x = f(t)$ and $y = g(t)$ describing motion along the hyperbola $-\frac{x^2}{16} + \frac{y^2}{4} = 1$, such that the point (x, y) is at $(0, -2)$ when $t = 0$ and it moves to the right as t increases.

ANSWER:

We recall that $\sec t$ and $\tan t$ satisfy the identity $\sec^2 t = 1 + \tan^2 t$, or equivalently $\sec^2 t - \tan^2 t = 1$. If we let $\sec^2 t = \frac{y^2}{4}$ and $\tan^2 t = \frac{x^2}{16}$ then the equation for the hyperbola will be satisfied. That means we can use $x = \pm 4 \tan t$ and $y = \pm 2 \sec t$, where we still have to choose the signs.
Putting in $t = 0$ has to give us $x = 0$ and $y = -2$: Since $\sec 0 = 1$, this forces us to pick the $-$ sign for y, $y = -2 \sec t$. Now as t increases from 0, $\tan t$ also increases. We want x to be increasing, so that the point moves to the right, so we have to choose the $+$ sign for x. Hence the parametric equations are $x = 4 \tan t$ and $y = -2 \sec t$.

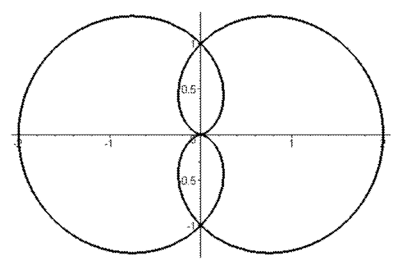
Problem 6



(a) For the curve $r = 4\sin\ 3\theta$, find the slope where $\theta = \frac{\pi}{4}$. The plot to the right shows roughly what this curve looks like: You must calculate the slope using derivatives to get credit.

ANSWER:
We use the formula $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$ with $f(\theta) = r = 4\sin 3\theta$. Then $f'(\theta) = 12\cos 3\theta$. At the point where $\theta = \frac{\pi}{4}$: $\sin\theta = \cos\theta = \frac{\sqrt{2}}{2}$, $f(\theta) = 4\sin 3\theta = 4\sin\frac{3\pi}{4} = 2\sqrt{2}$, and $f'(\theta) = 12\cos 3\theta = 12\cos\frac{3\pi}{4} = -6\sqrt{2}$.

Putting those numbers into the formula we note that the $\sin\theta$ and $\cos\theta$ factors in the numerator and denominator are all $\frac{\sqrt{2}}{2}$ and so they cancel out. That leaves us with $\frac{dy}{dx} = \frac{f'(\theta)+f(\theta)}{f'(\theta)-f(\theta)} = \frac{-6\sqrt{2}+2\sqrt{2}}{-6\sqrt{2}-2\sqrt{2}} = \frac{-4}{-8} = \frac{1}{2}$.



(b) Find the points where $r = 1 + \cos\ \theta$ and $r = 1 - \cos\ \theta$ intersect. You can use the plot at the right as a check of your work but you must show how you calculate the specific coordinates of each intersection point: Just reading the intersection points from the plot will not receive any credit.

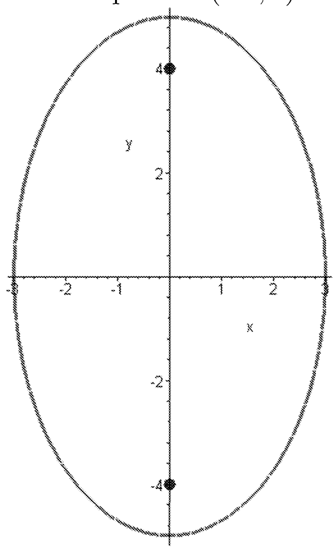
ANSWER:
First we try setting the two r values equal. We get the equation $1+\cos\theta = 1-\cos\theta$ or $2\cos\theta = 0$, so $\cos\theta = 0$. That occurs for $\theta = \pm\frac{\pi}{2}$. We put that into each function and find they both give $r = 1 - 0 = 1$ so each curve passes through the point $(1, \frac{\pi}{2})$ and through the point $(1, -\frac{\pi}{2})$, confirming the two points of intersection the picture shows on the upper and lower y-axis.

Now it appears both curves go through the origin which we think of as $(0,0)$, but that does not work in either equation! But $r = 1 + \cos\theta$ takes on the value 0 when $\cos\theta = -1$, e.g. when $\theta = \pi$, and $r = 1 - \cos\theta$ gives 0 when $\cos\theta = 1$, e.g. when $\theta = 0$, so the origin is on each curve, just for different θ values. So the origin, whether labelled $(0,0)$ or $(0,\pi)$, is an intersection point for the curves.

Problem 7

(a) Sketch the curve $\frac{x^2}{9} + \frac{y^2}{25} = 1$. Be sure to show where it crosses the x-axis and/or the y-axis (write out the coordinates!), and where its foci are (write out the coordinates!). Your sketch will not be graded for drawing ability but should resemble the correct curve.

ANSWER:
Here is Maple's plot of this ellipse. The curve crosses the x-axis where $y = 0$, so $x^2 = 9$ and $x = \pm 3$. Similarly the y-intercept is where $y = \pm 5$. In our usual notation a is the larger of those sizes, 5, and b is the smaller, 3, so the distance from the center to a focus is $\sqrt{25 - 9} = 4$. Hence the intercepts are $(\pm 3, 0)$ and $(0, \pm 5)$, and the foci are at $(0, \pm 4)$.



(b) The equation $4x^2 + 2\sqrt{3}\,xy + 2y^2 + 10\sqrt{3}\,x + 10y = 5$ describes a conic section. (An ellipse, parabola, or hyperbola, not a degenerate case such as a line or point.)

(i) Find an angle θ such that rotation of the coordinate system by θ would eliminate the xy term. (You do not need to carry out the rotation!)

ANSWER:
This equation has the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ where $A = 4$, $B = 2\sqrt{3}$, $C = 2$, and the rest don't matter for this problem. We can rotate by any angle θ that satisfies $\cot 2\theta = \frac{A-C}{B} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$. If $\cot 2\theta = \frac{1}{\sqrt{3}}$, $\tan 2\theta = \sqrt{3}$. So one choice is $2\theta = \frac{\pi}{3}$, in which case $\theta = \frac{\pi}{6}$.

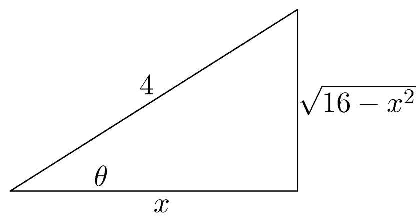
(ii) Which kind of curve (ellipse, parabola, or hyperbola) is this conic section?

ANSWER:
We can use the discriminant $B^2 - 4AC = (2\sqrt{3})^2 - 4 \times 4 \times 2 = 12 - 32 = -20$. Since that is negative, the curve is an ellipse.

Problem 8

(a) Evaluate the integral $\int \frac{x^3\,dx}{\sqrt{16-x^2}}$.

ANSWER:
It is possible to do this with a "u substitution" that does not involve trigonometry, but the difference of squares suggests a trig substitution so I will do it that way. If we draw a triangle and label it as



we are led to $x = 4\cos\theta$ so $dx = -4\sin\theta$, and $\sqrt{16-x^2} = 4\sin\theta$. Thus the integral becomes $\int \frac{(4\cos\theta)^3}{4\sin\theta}(-4\sin\theta)d\theta = -\int 64\cos^3\theta\,d\theta$. We separate $\cos^3\theta\,d\theta$ as $(\cos^2\theta)\cos\theta\,d\theta$ and use $\cos^2\theta = 1-\sin^2\theta$, and have $-64\int(1-\sin^2\theta)\cos\theta\,d\theta = -64\int\cos\theta\,d\theta + 64\int\sin^2\theta\cos\theta\,d\theta$. We use the substitution $u = \sin\theta$ on the second integral and get $-64\sin\theta + \frac{64}{3}\sin^3\theta + C$. From the triangle we read $\sin\theta = \frac{\sqrt{16-x^2}}{4}$ and substituting that in gives $-16\sqrt{16-x^2} + \frac{1}{3}(16-x^2)^{\frac{3}{2}} + C$.

(b) One of the integrals $\int_1^\infty \frac{dx}{\sqrt{x}}$ and $\int_1^\infty \frac{dx}{x^3}$ converges, and the other does not. Evaluate the one that converges.

ANSWER:
We could try evaluating each to see which one converges. But each is of the form $\int_1^\infty \frac{dx}{x^p}$, which we know converges only if $p > 1$, whereas we have to choose between $p = \frac{1}{2}$ and $p = 3$. So we go with $p = 3$ and evaluate $\int_1^\infty \frac{dx}{x^3}$.

To evaluate this improper integral we need to set it up as a limit, $\lim_{b\to\infty}\int_1^b \frac{dx}{x^3}$. The integration (power rule) yields $\frac{1}{-2}x^{-2}\Big]_1^b = \frac{1}{-2b^2} - \frac{1}{-2} = \frac{1}{2}\left(1-\frac{1}{b^2}\right)$. Taking the limit as $b\to\infty$ we get $\frac{1}{2}$ as the answer.