Your Name: ____________________________________

Circle your TA's name:

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Mathematics 222, Fall 2001 — **Lecture 1 (Wilson)**

Final Exam December 21, 2001

Write your answers to the twelve problems in the spaces provided. If you must continue an answer somewhere other than immediately after the problem statement, be sure (a) to tell where to look for the answer, and (b) to label the answer wherever it winds up. In any case, be sure to make clear what is your final answer to each problem.

Wherever applicable, leave your answers in exact forms (using π, $\sqrt{3}$, $\cos^{-1}(0.6)$, and similar numbers) rather than using decimal approximations. If you use a calculator to evaluate your answer be sure to show what you were evaluating!

You may refer to one or two pages of notes you have brought with you, as announced in class.

BE SURE TO SHOW YOUR WORK, AND EXPLAIN WHAT YOU DID. YOU MAY RECEIVE REDUCED OR ZERO CREDIT FOR UNSUBSTANTIATED ANSWERS. ("I did it on my calculator" and "I used a formula from the book" are not sufficient substantiation...)

Problem	Points	Score
1	14	
2	15	
3	24	
4	18	
5	14	
6	16	
7	24	
8	15	
9	15	
10	15	
11	15	
12	15	
TOTAL	200	

Some formulas, identities, and numeric values you might find useful:

Values of trig functions:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	—

Trig facts:

1. $\tan\theta = \frac{\sin\theta}{\cos\theta}$
2. $\sec\theta = \frac{1}{\cos\theta}$
3. $\sin^2\theta + \cos^2\theta = 1$
4. $\sec^2\theta = \tan^2\theta + 1$
5. $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
6. $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
7. $\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$
8. $\sin^2 x = \frac{1}{2}(1-\cos 2x)$
9. $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

Derivative formulas:

1. $\frac{d}{dx}\tan x = \sec^2 x$
2. $\frac{d}{dx}\sec x = \sec x\,\tan x$
3. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
4. $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
5. $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$
6. $\frac{d}{dx}\ln x = \frac{1}{x}$
7. $\frac{d}{dx} e^x = e^x$

Integral formulas: (We assume you know, and you are certainly allowed to use, basic formulas for integrals of functions such as x^n, e^x, $\sin x$, $\cos x$, etc., and how to use substitution to extend these.)

1. $\int \frac{1}{u}\,du = \ln|u| + C$
2. $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$
3. $\int \frac{du}{1+u^2} = \tan^{-1} u + C$
4. $\int \sec(u)\,du = \ln|\sec(u) + \tan(u)| + C$
5. $\int u\,dv = uv - \int v\,du$

Algebra formulas:

1. $\ln(xy) = \ln(x) + \ln(y)$
2. $a^{x+y} = a^x\,a^y$
3. $a^b = e^{b\ln a}$

Problem 1 (14 points)
An ellipse has one focus at $(2,0)$ and the other focus at $(8,0)$. One place it crosses the axis is at the origin. Find an equation for this ellipse.

Problem 2 (15 points)
For the polar curve $r = 2\sin 3\theta$, find the angle between the radius vector and the tangent to the curve at $(2, \frac{\pi}{6})$.

Problem 3 (24 points)
Evaluate the integrals:

(a) $$\int e^x \sin(2x)\, dx$$

(b) $$\int_{-1}^{1} \frac{dx}{x^{2/3}}$$

(The correct numeric answer for this integral will receive no credit unless you show a correct procedure leading to the answer.)

(c) $$\int \frac{x^2\, dx}{\sqrt{16 - x^2}}$$

Problem 4 (18 points)
Let $A = (2, -1, 3)$, $B = (3, 1, 2)$, and $C = (4, -1, 4)$.

(a) Find the cosine of the angle between the line through A and B and the line through B and C.

(b) Find an equation for the plane containing the three points A, B, and C.

(c) Find equations for the line through B which is perpendicular to the plane you found in (b).

Problem 5 (14 points)
Let $\vec{u} = 2\vec{\imath} + \vec{\jmath} + 0\,\vec{k}$ and $\vec{v} = \vec{\imath} + 2\vec{\jmath} + 0\,\vec{k}$.
What is the direction of $\vec{u} \times \vec{v}$?
(You do not have to compute $\vec{u} \times \vec{v}$ to answer this problem. You will get some of the credit for this problem if you correctly describe a line along which $\vec{u} \times \vec{v}$ lies. To get all the credit you must tell which direction it points along that line.)

Problem 6 (16 points)
While studying for your math exam you gradually get tired. Each hour you are only 90% as effective as in the previous hour. So if we call the amount you learned in the first hour 1 unit, then in the second hour you learn 0.9 units, in the third hour $0.9 \times 0.9 = 0.81$ units, etc.

(a) When you have studied for 10 hours total, how many units have you learned? (Give an algebraic expression for this answer, not just a number from a calculator.)

(b) If you studied forever with this same declining effectiveness, how much would you learn?

Problem 7 (24 points)

For each series: Tell whether it converges absolutely, converges conditionally, or does not converge at all. Be sure to give reasons for your answers.

(a) $$\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^3+1}$$

(b) $$\sum_{n=1}^{\infty}(-1)^n\frac{n}{2n+1}$$

(c) $$\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2-3}$$

Problem 8 (15 points)
Solve the initial value problem:

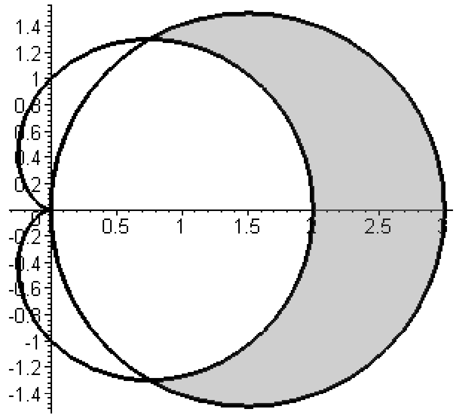
$$e^y \frac{dy}{dx} = \cos x \qquad \text{and} \qquad y(0) = 1$$

Problem 9 (15 points)
Solve the differential equation:

$$\frac{dy}{dx} - y \tan x = 1 \qquad \text{(you may assume } -\frac{\pi}{2} < x < \frac{\pi}{2}\text{)}$$

Problem 10 (15 points)
Let $f(x) = \ln(3x + 1)$.
Write the beginning of the Maclaurin series for $f(x)$. Show all terms with degree less than or equal to four. You do not need to find a formula for the n^{th} degree term in general.

Problem 11 (15 points)



Find the area of the region which is inside the circle $r = 3\cos\theta$ but outside the cardioid $r = 1 + \cos\theta$.

Be sure to show your work in detail. Just looking at the picture and guessing where a point is, for example, will not get credit.

Problem 12 (15 points)
Find the general solution to the differential equation:

$$y'' + y' - 2y = \sin x.$$

SCRATCH PAPER