Mathematics 221, Summer 2009 **(Wilson)**

Final Exam August 6, 2009
Answers to both morning and afternoon parts

Problem 1 (16 points)
Find the derivative $\frac{dy}{dx}$ for each of the following functions:

(a) $y = \ln(\sin(x))$

Answer: Using the chain rule, $\frac{dy}{dx} = \frac{1}{\sin x} \times \cos x = \frac{\cos x}{\sin x} = \cot x$. (Any of those forms would be an acceptable answer.)

(b) $y = e^{x^2+2}$

Answer: Again we need the chain rule, $\frac{dy}{dx} = \left(e^{x^2+2}\right) \times (2x) = 2xe^{x^2+2}$.

(c) $y = (\tan^{-1} x)(\sin x)$ (equivalently, $y = (\arctan x)(\sin x)$)

Answer: Using the product rule we get $\frac{dy}{dx} = \frac{1}{1+x^2} \times \sin x + (\tan^{-1} x)(\cos x) = \frac{\sin x}{1+x^2} + (\tan^{-1} x)(\cos x)$.

(d) $y = \sin^{-1}(\sqrt{x})$ (equivalently, $y = \arcsin(\sqrt{x})$)

Answer: $\frac{dy}{dx} = \frac{1}{\sqrt{1-x}} i \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}$.

Problem 2 (13 points)
Using the definition of the derivative as a limit, find the derivative $\frac{df}{dx}$ for $f(x) = 3x^2 - 2x + 1$.
Answer:

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(3(x+h)^2 - 2(x+h) + 1) - (3x^2 - 2x + 1}{h} \\
&= \lim_{h\to 0} \frac{(3x^2 + 6xh + h^2 - 2x - 2h + 1) - (3x^2 - 2x + 1)}{h} \\
&= \lim_{h\to 0} \frac{6xh + h^2 - 2h}{h} \\
&= \lim_{h\to 0} \frac{h(6x + h - 2)}{h},
\end{aligned}$$

but in taking that limit we don't need to include the case $h = 0$ so we can cancel h out of numerator and denominator, so we can continue with

$$= \lim_{h\to 0}(6x + h - 2). \qquad (1)$$

The limit of h as $h \to 0$ is 0, while the other two terms are constants so far as h is concerned, so that limit gives us $\frac{df}{dx} = 6x - 2$.

Problem 3 (14 points)

Let $f(x) = e^{\sin 2x}$.

(a) Find an equation for the tangent line to the graph of $f(x)$ at the point $(0, 1)$.

Answer: The slope of the tangent line will be $f'(x)$ evaluated at $x = 0$: $f'(x) = e^{\sin x}(2\cos 2x)$, so $f'(0) = e^0 \times 2 \times 1 = 2$. Since the line passes through $(0, 1)$ with slope 2, it must have as one form for an equation $y - 1 = 2(x - 0)$ which we can rearrange as $y = 2x + 1$.

(b) Use the tangent line to find an approximate value for $f(0.1)$.

Answer: We find the value of y on the tangent line for $x = 0.1$, i.e. $y = 2(0.1) + 1 = 1.2$. (Maple gives the value of $f(0.1)$ as approximately 1.219778556 so this is not too bad!)

Problem 4 (20 points)

The function $f(x) = 3x^4 + 4x^3 - 12x^2$ has critical points where $f'(x) = 0$ at $x = -2$, $x = 0$, and $x = 1$. Find all absolute and local maxima and minima for $f(x)$ on the interval $-3 \leq x \leq 2$.
For each point you identify: (a) Tell whether it gives an absolute maximum, an absolute minimum, a relative maximum, and/or a relative minimum. (b) Tell how you know your answer to (a). (c) Give both the x-value and the resulting value of $f(x)$.
Answer: Although we are already given the points where $f'(x) = 0$, it will be useful to know f' and f'' when testing for max or min: $f'(x) = 12x^3 + 12x^2 - 24x$, and $f''(x) = 36x^2 + 24x - 24$.
Now we consider one-by-one the points that are either critical points or end points of the interval. Here is a table to help organize the results:

Which point	why	other info
$x = -3$	left endpoint	$f'(-3) = 12(-27) + 12(9) + 72 < 0$
$x = -2$	critical point	$f''(-2) = 36(4) + 24(-2) - 24 > 0$
$x = 0$	critical point	$f''(0) = -24 < 0$
$x = 1$	critical point	$f''(1) = 36 + 24 - 24 > 0$
$x = 2$	right endpoint	$f'(2) = 12(8) + 12(4) - 48 > 0$

Analyzing for local max/min: Since f is decreasing at the left endpoint $x = -3$, that point gives at least a local maximum. Since f'' is positive at $x = -2$ where $f'(x) = 0$, the second derivative test says there is a local minimum there. Since f'' is negative at $x = 0$ where $f'(x) = 0$, the second derivative test says there is a local maximum there. Since f'' is positive at $x = 1$ where $f'(x) = 0$, the second derivative test says there is a local minimum there. Since f is increasing at the right endpoint $x = 2$, that point gives at least a local maximum.
Computing the values of the function itself at the five points we get $f(-3) = 3(81) + 4(-27) - 12(9) = 27$, $f(-2) = 3(16) + 4(-8) - 12(4) = -32$, $f(0) = 0$, $f(1) = 3 + 4 - 12 = -5$, and $f(2) = 3(16) + 4(8) - 12(4) = 32$. So the absolute maximum is at $x = 2$ and the absolute minimum is at $x = -2$.
Collecting the results: $(-3, 27)$ is a local maximum; $(-2, -32)$ is both a local and and an absolute minimum; $(0, 0)$ is a local maximum; $(1, -5)$ is a local minimum; $(2, 32)$ is both a local and an absolute maximum.

Problem 5 (18 points)
Evaluate the following integrals:

(a) $\int_{-1}^{1} 2x\cos(x^2-2)\,dx$

Answer: If we let $u = x^2 - 2$ we see that its derivative $2x$ is there for us to use. So with that choice of u, $du = 2x\,dx$. When $x = -1$, $u = 1^2 - 2 = -1$, and when $x = 1$ we also get $u = -1$. So the integral becomes $\int_{-1}^{-1} \cos(u)\,du$. You can proceed further with that, but since the upper and lower endpoints are the same the answer must be 0. In fact you could have noticed to begin with that the function being integrated is an odd function: the $-2x$ part changes sign but not size when x is replaced by $-x$, while x^2 and hence the rest of the function does not change at all. So we have the integral of an odd function along an interval symmetric about 0 and we had a theorem saying that must give 0.

(b) $\int \frac{x-2}{(x^2-4x+3)^3}\,dx$

Answer: Again we look for a substitution: If we let $u = x^2 - 4x + 3$, that both provides us something we can"build" the derivative for ($\frac{du}{dx} = 2x - 4$ is exactly twice the numerator) and which will simplify the problem if we make that substitution. We have $du = (2x-4)dx = 2(x-2)dx$ so we get $\frac{1}{2}\int \frac{du}{u^3} = \frac{1}{2}\int u^{-3}\,du = \frac{1}{2}\left(\frac{u^{-2}}{-2}\right) + C = -\frac{1}{4u^2} + C = -\frac{1}{4}\frac{1}{(x^2-4x+3)^2} + C.$

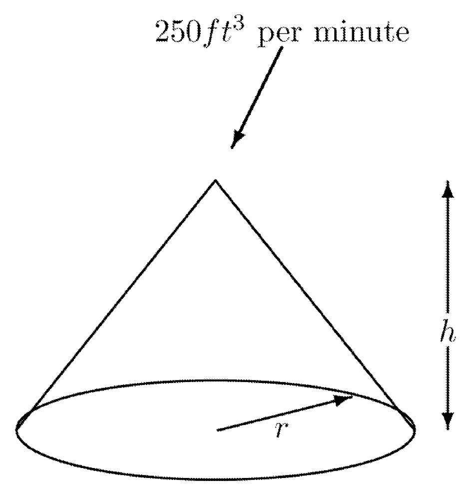
(c) $\int_0^{\frac{\pi}{2}} \sin^2(x)\cos(x)\,dx$

Answer: Letting $u = \sin x$, $du = (\cos x)dx$. When $x = 0$, $u = sin(0) = 0$, and when $x = \frac{\pi}{2}$, $u = \sin(\frac{\pi}{2}) = 1$, so the integral becomes $\int_0^1 u^2 du = \left.\frac{u^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$.

Problem 6 (12 points)
Solve the initial value problem: $\frac{dy}{dt} = \sin(t) + \cos(t)$, and $y\left(\frac{\pi}{2}\right) = 4$.
Answer: Antidifferentiating, y must be of the form $\int(\sin t + \cos t)dt = -\cos t + \sin t + C$, i.e. $y = -\cos t + \sin t + c$ for some constant c. Putting in the known value, $y(\frac{\pi}{2}) = -\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + c = 0 + 1 + c$, but we are given that $y(\frac{\pi}{2}) = 4$, so $1 + c = 4$ and hence $c = 3$.
Thus the function must be $y(t) = -\cos t + \sin t + 3$ in order to satisfy both of the conditions.

Problem 7 (20 points)

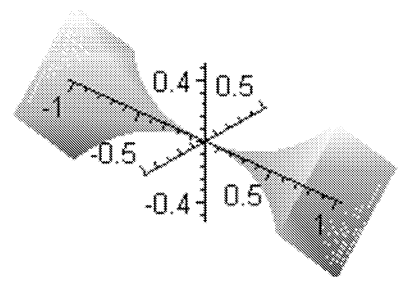
Sand is being added to a conical pile at a constant rate of 250 cubic feet per minute. The sand always flows out so that the height of the cone is the same as the radius of the circular base of the cone. At the instant when the radius is 5 feet, how fast is the radius increasing?



Answer: If we let $V(t)$ denote the volume of sand at time t, $r(t)$ the radius of the circular base, and $h(t)$ the height of the cone, we have $V = \frac{1}{3}h(\pi r^2)$ but since $h = r$ that simplifies to $V = \frac{\pi}{3}r^3$. Differentiating with respect to t and remembering the chain rule we have $\frac{dV}{dt} = \frac{\pi}{3}(3r^2)\frac{dr}{dt} = \pi r^2 \frac{dr}{dt}$. The sand added to the pile tells us that the volume is increasing at 250 cubic feet per minute, i.e. $\frac{dV}{dt} = 250$. We want $\frac{dr}{dt}$ at the instant when $r = 5$, which must satisfy $250 = \pi(5^2)\frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{250}{25\pi}$ $= \frac{10}{\pi}$ feet per minute.

Problem 8 (16 points)

A solid object extends from $x = -1$ to $x = 1$. If we cut across the x-axis at any point in that range, the cross section is a square whose diagonal is $3\,x^2$. What is the volume of this object?



Answer: If the cross section at x is a square whose diagonal is $3x^2$, the length of a side of the square is $3x^2/\sqrt{2}$ so the area of that square cross section is $\left(3x^2/\sqrt{2}\right)^2 = \frac{9}{2}x^4$. You might remember how to proceed using a forumula from section §6.1, but we can easily derive it by thinking of slices across the x-axis of thickness Δx: Each slice has area on one side $\frac{9}{2}x^4$, so its volume is approximately $\frac{9}{2}x^4\Delta x$. The slices extend from $x = -1$ to $x = 1$. Adding those up and taking a limit as the slices get thinner, we have $\displaystyle\int_{-1}^{1} \frac{9}{2}x^4 dx = \frac{9}{2}\frac{x^5}{5}\Bigg]_{-1}^{1} = \frac{9}{2}\left(\frac{1}{5} - (-\frac{1}{5})\right) = \frac{9}{2}\frac{2}{5} = \frac{9}{5}$ cubic units.

Problem 9 (12 points)

Let $f(x) = \displaystyle\int_{-1}^{x} \sec(t)\,dt.$

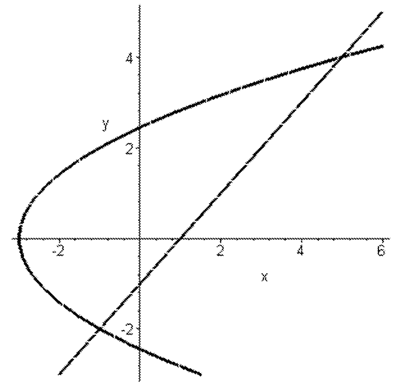
(a) What is $\frac{df}{dx}$?

Answer: Since we are taking the derivative with respect to exactly the variable at the upper limit of integration, the Fundamental Theorem of Calculus (part 1) says we can get the derivative by simply "plugging in" x into the function being integrated. So $\frac{df}{dx} = \sec(x)$.

(b) What is $\frac{d^2f}{dx^2}$?

Answer: Now that we know $\frac{df}{dx}$, we can get the second derivative by simply differenting the first derivative: $\frac{d^2f}{dx^2} = \frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$.

Problem 10 (20 points)



Find the area of the region in the plane bounded by $y = x - 1$ and $y^2 = 2x + 6$. (The region has the shape shown at the right.)

Answer: First we need to locate those "corners" where the two graphs meet: At those points the (x, y) values for both functions must agree, so y^2 from one must match y^2 from the other. Hence $(x-1)^2 = 2x+6$, $x^2 - 2x + 1 = 2x + 6$, $x^2 - 4x - 5 = 0$: You could use the quadratic formula, or factor as $(x-5)(x+1) = 0$, and either way we get $x = -1$ and $x = 5$. Using $y = x - 1$ we see the two corner points are $(-1, -2)$ and $(5, 4)$. The parabola $y^2 = 2x + 6$ crosses the x axis, determining the left-most point, at $y = 0$ and hence $2x + 6 = 0$ or $x = -3$.
We could elect to find the area using either an integral with "dx" or "dy". If we use "dx", coming from slices across the x-axis which will be vertical slices, we see that slices to the left of $x = -1$ have their top and bottom ends both on the parabola $y^2 = 2x + 6$, while slices to the right of that line have their top ends on the parabola but their bottom ends determined by the straight line $y = x - 1$. So while that can be done, it will require evaluating two integrals, one for $-3 \le x \le -1$ and the other for $-1 \le x \le 5$, and adding the results.
So instead I will think of horizontal slices: At any y value between $y = -2$ and $y = 4$, the left end of a slice will be on the parabola and the right end on the straight line, so we can get by with just one integral. A slice, of thickness Δy, at height y, will have its left end at the x value that is on the parabola: From $y^2 = 2x + 6$ we see that is $2x = y^2 - 6$ or $x = \frac{1}{2}(y^2 - 6)$. The right end will be on the line $y = x - 1$ so $x = y + 1$. Hence the length of the slice from left to right is $(y+1) - \frac{1}{2}(y^2 - 6)$ which simplifies to $4 + y - \frac{y^2}{2}$. So the area the slice contributes is $\left(4 + y - \frac{y^2}{2}\right)\Delta x$: Adding those contributions and getting an integral as the limit for narrower slices, we have the area as

$$\int_{-2}^{4} \left(4 + y - \frac{y^2}{2}\right) dy = \left(4y + \frac{y^2}{2} - \frac{y^3}{6}\right)\Bigg]_{-2}^{4} = \left(16 + \frac{16}{2} - \frac{64}{6}\right) - \left(-8 + \frac{4}{2} + \frac{8}{6}\right) = 18.$$

Problem 11 (21 points)
Evaluate the integrals:

(a) $\displaystyle\int \frac{1}{\sqrt{1 - 4x^2}} dx$

Answer: This looks a lot like the integral that gives us an inverse sine, but we have $4x^2$ instead of x^2. Substitute $u = 2x$, $du = 2dx$, and we have $\displaystyle\int \frac{1}{\sqrt{1-u^2}} du = \frac{1}{2}\sin^{-1} u + C$, which in terms of the original variable is $\frac{1}{2}\sin^{-1} 2x + C$.

(b) $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x}{1 + e^{2x}} dx$

Answer: (You might choose to rewrite $\ln\sqrt{3}$ as $\frac{1}{2}\ln 3$: That is OK, but in fact the rest of the arithmetic is slightly easier without that "simplification".)

The e^x in the numerator is also the derivative of e^x, and the denominator contains $e^{2x} = (e^x)^2$, so we can let $u = e^x$ with $du = e^x dx$. When $x = 0$, $u = e^0 = 1$. When $x = \ln\sqrt{3}$, $u = e^{\ln\sqrt{3}} = \sqrt{3}$. So the integral becomes $\int_1^{\sqrt{3}} \frac{1}{1+u^2} du = \tan^{-1} u\Big]_1^{\sqrt{3}}$. Now $\tan(\frac{\pi}{3}) = \sqrt{3}$, so $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$. Similarly, $\tan(\frac{\pi}{4}) = 1$, so $\tan^{-1}(1) = \frac{\pi}{4}$. Thus the integral works out to $\frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$.

(c) $\int_{e^2}^{e^4} \frac{1}{x \ln x} dx$

<u>Answer:</u> If we let $u = \ln x$, $du = \frac{1}{x} dx$. When $x = e^2$, $u = \ln e^2 = 2$, and when $x = e^4$, $u = \ln e^4 = 4$. So the integral becomes $\int_2^4 \frac{1}{u} du = \ln(u)]_2^4 = \ln(4) - \ln(2) = 2\ln 2 - \ln 2 = \ln 2$.

<u>Problem 12</u> (18 points)

Let $f(x) = 2x^3 - 3x^2 - 5x + 4$.

(a) At $x = 0$, is $f(x)$ increasing or decreasing? How can you tell?

<u>Answer:</u> $f'(x) = 6x^2 - 6x - 5$, so $f(0) = -5$. The function is decreasing since its derivative is negative.

(b) At $x = 0$, is the graph of $f(x)$ concave upward or concave downward? How can you tell?

<u>Answer:</u> $f''(x) = 12x - 6$, so $f''(0) = -6$. Since the second derivative is negative, the graph is concave downward.

(c) At what value of x does the graph switch from concave one way to concave the opposite way? (I.e., where is there a point of inflection?)

<u>Answer:</u> The concavity will switch where the second derivative changes sign. The second derivative, $12x - 6$, is continuous so it can only change sign where it goes through zero, where $12x - 6 = 0$, i.e. $x = \frac{1}{2}$. To the left of $x = \frac{1}{2}$ the second derivative is negative so the graph is concave downward (as for example in part (b)) and to the right of $x = \frac{1}{2}$ the concavity is upward.