Math 221, Summer 2009

Answers to First Midterm Exam July 9, 2009

Problem 1:

Suppose $f(t) = t^3 + 3t^2 - 72t + 4$ gives the position, in inches along some scale, of an object, at time t in seconds.

(a) What is the average rate of change of $f(t)$ as t goes from $t = -1$ to $t = 1$? (Remember units!)

Answer: The position at $t = -1$ is $f(-1) = 78$, and the position at time $t = 1$ is $f(t) = -64$. So the average rate of change, the average velocity of the object over that time interval, is $\frac{-64-78}{1-(-1)} = -71$ inches per second.

(b) What is the instantaneous rate of change of $f(t)$ at $t = 0$? (Remember units!)

Answer: $f'(t) = 3t^2 + 6t - 72$ so $f'(0) = -72$ inches per second.

(c) Find all local maxima and minima of $f(t)$. Be sure to indicate whether a given point gives a maximum or gives a minimum.

Answer: We look for critical points. f' as found in (b) exists for any x, and we have no closed interval, so the places to look are where the derivative is zero. $f'(t) = 3(t^2 + 2t - 24) = 3(t-4)(t+6)$, so the significant t values are $t = 4$ and $t = -6$. If we take the second derivative, $f''(t) = 6t + 6$, so $f''(4)$ is positive and $f''(-6)$ is negative. Using the second derivative test we see there is a local maximum at $t = -6$, where $f(t) = 328$, and a local minimum at $t = 4$, where $f(t) = -172$.

(d) $f(t)$ would not have to have absolute minimum and maximum values on the interval $-10 < t < 10$. Give an example of an interval on which you know $f(t)$ would have to have absolute maximum and minimum values, and tell why you know that.

Answer:

The problem is that $(-10, 10)$ is not a closed interval. It does happen that this f does have an absolute minimum on that interval at $t = 4$, because it happens that -172 is lower than any other value of f on the interval. But if we just made it a closed interval, or for that matter used absolutely any closed interval, as a continuous function it must have absolute maximum and minimum values on the interval. So I will just change it to $-10 \le t \le 10$, and the theorem tells us that the continuous function together with this closed interval will work.

Problem 2:

Evaluate the derivatives $f'(x)$ for:

(a) $f(x) = 3x^2 + \frac{4}{x^3}$.

Answer: We could use the quotient rule on the second term, or we can rewrite it as a power: $f(x) = 3x^2 + 4x^{-3}$. If we use this form, $f'(x) = 6x + 4(-3)x^{-4} = 6x - \frac{12}{x^4}$. The quotient rule will give the same result, except that there are various algebraically equivalent ways to express this same answer.

(b) $f(x) = \tan(3x^2 + 2)$.

Answer: The derivative of the tangent is the square of the secant, but we need to remember the chain rule! $f'(x) = 6x\sec^2(3x^2 + 2)$.

(c) $f(x) = (x^2 + 1)\cos(4x - 1)$.

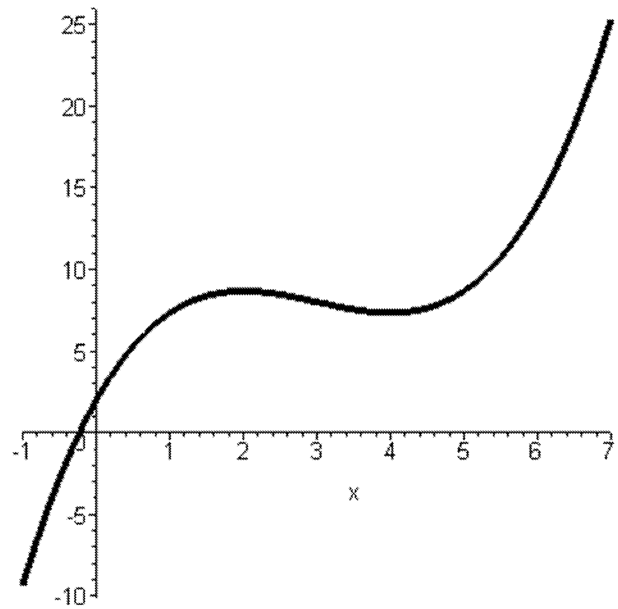
Answer: Using the product rule and the chain rule, and the derivative of cosine is the negative of the sine: $f'(x) = (2x)\cos(4x - 1) - 4(x^2 + 1)\sin(4x - 1)$.

(d) $f(x) = \dfrac{x^2 \sin(x)}{\tan(x)}$.

Answer: The "obvious" way to do this is either to use the quotient rule or write the denominator in the numerator to the (-1) power. But there is an easier way: $\tan x = \frac{\sin x}{\cos x}$, so $f(x) = (x^2 \sin(x))/\frac{\sin x}{\cos x} = x^2 \cos x$. Now we just need the product rule, $f'(x) = 2x\cos(x) - x^2 \sin(x)$.

Problem 3:

The picture at the right shows part of the graph of a continuous function $f(x)$. Assume that if we continued the graph to the left and the right no "new" behavior would be found, that the slopes and curvatures continue from what you can see. Answer the questions below based on this function.



(a) $f'(x) = 0$ at $x =$ (List all numbers that make that statement true.)

Answer: We need the places where the tangent line would be horizontal, $x = 2$ and $x = 4$.

(b) $f''(x) = 0$ at $x =$ (List all numbers that make that statement true.)

Answer: The second derivative changes from negative to positive at $x = 3$, since the curve changes from concave down to concave up. So if it is continuous it must be zero at $x = 3$.

(c) On the interval $-\infty < x < 2.5$, $f''(x)$ is $\quad < 0 \quad = 0 \quad > 0$. (Circle correct answer(s))

Answer: Anywhere from $-\infty$ to 2.5, the curve is concave downward. So $f''(x) < 0$.

(d) At $x = 3$, $f(x)$ is $\quad < 0 \quad = 0 \quad > 0$. (Circle correct answer(s))

Answer: At $x = 3$, the graph is above the x-axis. So $f(3) > 0$.

(e) At $x = 3$, $f'(x)$ is $\quad < 0 \quad = 0 \quad > 0$. (Circle correct answer(s))

Answer: At $x = 3$, the graph is sloping downward as we move to the right, so $f'(3) < 0$.

(f) At $x = 3$, $f''(x)$ is $\quad < 0 \quad = 0 \quad > 0$. (Circle correct answer(s))

Answer: At $x = 3$, the curve is changing from concave downward on the left to concave on the right, a point of inflection. So $f''(3) = 0$.

(g) On the interval $2 < x < 4$, $f(x)$ is $\quad < 0 \quad = 0 \quad > 0$. (Circle correct answer(s))

Answer: Everywhere on the interval from 2 to 4, the graph is above the x-axis. So $f'(x) > 0$ for all x values in that region.

(h) On the interval $5 < x < 6$, $f'(x)$ is $\quad < 0 \quad = 0 \quad > 0$. (Circle correct answer(s))

Answer: The section of the curve from $x = 5$ to $x = 6$ is sloping upward as we go to the right, so $f'(x) > 0$ at all x values in that region.

Problem 4:

(a) Find an equation for the tangent line to the graph of $y = \tan(x)$ at the point $(\frac{\pi}{4}, 1)$.

Answer: $y'(x) = \sec^2(x) = 1/\cos^2(x)$, so at $x = \frac{\pi}{4}$ the slope of the graph (and hence the tangent line) will be $1/\left(\frac{1}{\sqrt{2}}\right)^2 = 1/(\frac{1}{2}) = 2$. So the tangent line is the line through $\frac{\pi}{4}, 1$ with slope 2, which we can write as $y - 1 = 2(x - \frac{\pi}{4})$ or $y = 2x + (1 - \frac{\pi}{2})$.

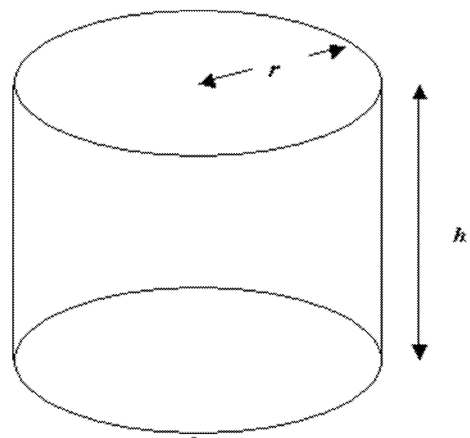
(b) Use a linear approximation to find an approximate value for $\tan(\frac{\pi}{4} - 0.1)$.

Answer: We want the y-value on the tangent line from (a), corresponding to $x = \frac{\pi}{4} - 0.1$. We can be formal and let $L(x) = 2x + (1 - \frac{\pi}{2})$, and evaluate $L(\frac{\pi}{4} - 0.1)$, but that formality is not essential. We get $L(\frac{\pi}{4} - 0.1) = 2(\frac{\pi}{4} - 0.1) + (1 - \frac{\pi}{2}) = \frac{\pi}{2} - 0.2 + 1 - \frac{\pi}{2} = 0.8$. (The actual value of $\tan(\frac{\pi}{4} - 0.1)$ is about 0.817776. This is not a terribly accurate approximation. Why? y'' at this point is 4, so the graph would be quite concave upward and hence curving away from the straight tangent line both to the left and right.)

Problem 5:

A metal can is to be made in the form of a right circular cylinder, to hold exactly one liter (1000 cubic centimeters) of some product. What are the dimensions (radius, height) in centimeters the cylinder should have in order to minimize the amount of metal required? (The amount is the area, we will assume the thickness has already been determined.)

Answer: First I will label the measurements of the cylinder as shown at the right, r for the radius and h for the height. The volume V will be the area of the base, πr^2, times the height, i.e. $V = \pi r^2 h$, so we know $\pi r^2 h = 1000$ in cubic centimeters. The metal of the can is in (a) two circular pieces on the top and bottom, each with area πr^2, and (b) the outside which amounts to a rectangle h tall and $2\pi r$ long if we unroll it. So the area of the metal is $2 \times \pi r^2 + 2\pi r h$. That is what we need to minimize. So, pulling things together, we need to minimize $2\pi r^2 + 2\pi r h$ where r and h satisfy $\pi r^2 h = 1000$.



If we solve $\pi r^2 h = 1000$ for h we get $h = \frac{1000}{\pi r^2}$, and if we substitute that for h in $2\pi r^2 + 2\pi r h$ we get the area A as a function depending only on r, $A(r) = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2}$ which simplifies to $A(r) = 2\pi r^2 + \frac{2000}{r}$. We look for the critical points of this function. Its derivative $A'(r) = 4\pi r - \frac{2000}{r^2}$ would not exist at $r = 0$, so we could include that as a critical point, but since $r = 0$ makes the volume also 0 that point would not satisfy our requirement that the volume is 1000 cc. and we can ignore it. So we turn to points where $A'(r) = 0$: There, $4\pi r - \frac{2000}{r^2} = 0$, $4\pi r = \frac{2000}{r^2}$, so $r^3 = \frac{2000}{4\pi} = \frac{500}{\pi}$. Now the second derivative $A''(r) = 4\pi + \frac{4000}{r^3}$ will be positive for any positive value of r, so there must be a local minimum at $r = \sqrt[3]{\frac{500}{\pi}}$.

So we want the radius to be $r = \sqrt[3]{\frac{500}{\pi}} = \left(\frac{500}{\pi}\right)^{\frac{1}{3}}$. Since we had already solved for $h = \frac{1000}{\pi r^2}$, the corresponding value for h is $1000/\left(\pi\left(\frac{500}{\pi}\right)^{\frac{2}{3}}\right)$. As decimal approximations those are $r \approx 5.419260701$ and $h \approx 10.83852140$ centimeters.

Problem 6:

Use the definition of the derivative as a limit to find $f'(x)$, for

$$f(x) = 3x^2 + 5x + 2.$$

Answer: There are a couple of forms in which we have seen this limit. I will use

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Working that out a piece at a time: $f(x+h) = 3(x+h)^2 + 5(x+h) + 2$, where $3(x+h)^2 = 3(x^2+2xh+h^2) = 3x^2+6xh+3h^2$, and $f(x) = 3x^2+5x+2$, so the numerator is $(3x^2+6xh+3h^2+5x+5h+2) - (3x^2+5x+2) = 6xh+3h^2+5h$. That makes the limit $\lim_{h\to 0} \dfrac{6xh+3h^2+5h}{h}$ and since we don't need to consider $h = 0$ in that limit we can cancel h from numerator and denominator to get $\lim_{h\to 0}(6x+3h+5)$. As $h \to 0$ the first and last terms do not change, while the second goes to 0, so the limit, and hence the derivative, is $6x+5$.

Problem 7:

$$\text{If } x \text{ and } y \text{ satisfy } xy^2 + x^2y = \sin(y), \text{ find } \frac{dy}{dx}.$$

Answer: It does not look practical to solve that relation for y as a function of x or for x as a function of y, so we use implicit differentiation. Taking the derivative with respect to x of each term in the relation, using the product rule and chain rule, gives us

$$y^2 + 2xy\frac{dy}{dx} + 2xy + x^2\frac{dy}{dx} = \cos(y)\frac{dy}{dx}.$$

Moving all terms that do have $\frac{dy}{dx}$ to one side and all other terms to the other side we get

$$\frac{dy}{dx}\left(2xy + x^2 - \cos(y)\right) = -y^2 - 2xy,$$

and division then gives us

$$\frac{dy}{dx} = -\frac{y^2+2xy}{2xy+x^2-\cos(y)}.$$

Problem 8:
Use the ϵ–δ definition of $\lim_{x\to a} f(x)$ to justify the statement:

$$\lim_{x\to 3}(4x-2) = 10$$

Answer: Suppose we are given some $\epsilon > 0$. Since the graph of this function has slope 4, so any horizontal change produces four times that much vertical change, I will let $\delta = \frac{\epsilon}{4}$. Since ϵ was greater than zero, δ will be also. We need to show that this way of choosing δ "works" in the definition, which requires:

For any $\epsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x-3| < \delta$ it must follow that $|(4x-2)-10| < \epsilon$

For our choice of $\delta = \frac{\epsilon}{4}$, assume $0 < |x-3| < \delta$: We won't in fact need the first part, $0 < |x-3|$, but extra facts don't hurt anything. So we know $|x-3| < \delta = \frac{\epsilon}{4}$. That amounts to $-\frac{\epsilon}{4} < x-3 < \frac{\epsilon}{4}$: Multiply through by 4 and get $-\epsilon < 4x-12 < \epsilon$. Now rearrange $4x-12$ as $(4x-2)-10$, and we have $-\epsilon < (4x-2)-10 < \epsilon$ which is equivalent to $|(4x-2)-10| < \epsilon$, and we are through!