Mathematics 221, Spring 2006 Lecture 3 (Wilson)

Second Midterm Exam April 7, 2006 ANSWERS

Problem 1 (15 points)
Let $f(x) = 3x^4 - 4x^3 - 12x^2$, for $\infty \leq x \leq \infty$.

(a) Find all critical points of f.

ANSWER: Since f is given by a polynomial there will be no singular points, points where the derivative does not exist. Since the interval is the whole real line, there are no end points. Hence the only critical points are the stationary points, the places where the derivative is zero. Taking the derivative we get $f'(x) = 12x^3 - 12x^2 - 24x$. We can simplify that as $12(x^3 - x^2 + 2x)$ or $12x(x^2 - x + 2)$. That quadratic factors nicely, or you could use the quadratic formula, so $f'(x) = 12x(x-2)(x+1)$. Hence the only numbers that make the derivative zero are $x = 0$, $x = 2$, and $x = -1$, which must be the critical points.

(b) On what interval(s) of numbers is $f(x)$ increasing? On what interval(s) of numbers is $f(x)$ decreasing?

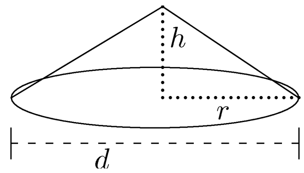
ANSWER: We look for where f' is positive and where it is negative. Since it is a polynomial $(12(x^3 - x^2 - 2x))$ it is continuous at all real numbers, hence it can change sign only by going through zero. In (a) we found that happens only at -1, 0, and 2. We consider the four parts those break the line into, $(-\infty, -1)$, $(-1, 0)$, $(0, 2)$, and $(2, \infty)$. On each of these the sign of the derivative cannot change, so we can check the sign at any point within that piece and it must apply everywhere on that piece. $f'(-2) = 12 \times (-8)$ is negative, so f' must be negative on all of $(-\infty, -1)$, so f is decreasing on that interval. On the second interval, $f'(-\frac{1}{2}) = 12 \times (-\frac{1}{8} - \frac{1}{4} + 1)$ is positive so f' is positive and f increasing on $(-1, 0)$. Moving along to the third piece, $f'(1) = 12 \times -2$ is negative, so f' must be negative and hence f decreasing everywhere on $(0, 2)$. On the last piece, $f'(3) = 12 \times 12$ which is positive, so using the same reasoning f' is positive and f is increasing on $(2, \infty)$.

(c) Find all (local or global) maxima and minima of f. Be sure to identify each point as to local or global, maximum or minimum.

ANSWER: We know the maxima and minima must occur at critical points, and we know those are -1, 0, and 2. In (b) we found that f' is changing from negative to positive as we pass $x = -1$, so that x value must produce a local minimum by the first derivative test. The actual minimum is the point $(x, f(x)) = (-1, f(-1)) = (-1, 3 + 4 - 12) = (-1, -5)$. Similarly the f' changes from positive to negative at $x = 0$, so there is a local maximum at $(0, 0)$. At $x = 2$, f' changes from negative to positive so there is a local minimum at $(2, f(2)) = (2, -32)$.

The function $f(x)$ grows without bound as $x \to \pm\infty$, so there is no global maximum. But the local minimum $(2, -32)$ has an f value less than the other local minimum, so it gives a global minimum.

Problem 2 (15 points)
Sand is pouring from a pipe at the rate of 16 cubic feet per second. If the falling sand forms a conical pile on the ground whose height h is always $\frac{1}{4}$ the diameter of the base of the pile, how fast is the height increasing when the pile is 4 feet high?
ANSWER: Since the radius is $\frac{1}{2}$ the diameter, and the diameter is 4 times the height, $r = \frac{d}{2} = \frac{4h}{2} = 2h$. Then the volume of sand in the pile, when the height is h, is $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3$. The volume is changing with time as 16 cubic feet of sand are added each second, so measuring in feet and seconds we can write $V'(t) = 16$. What we need to find is $h'(t)$ at the instant when $h = 4$.



Differentiating $V(t) = \frac{4}{3}\pi(h(t))^3$ with respect to t and remembering the chain rule, $V'(t) = \frac{4}{3}\pi \times 3(h(t))^2 \times h'(t) = 4\pi(h(t))^2 h'(t)$. Solving for h' we get $h'(t) = \dfrac{V'(t)}{4\pi(h(t))^2}$. At the instant we care about, $V'(t) = 16$ and $h(t) = 4$, so we get $h'(t) = \dfrac{16}{4\pi \times 16} = \dfrac{1}{4\pi}$ feet per second.
(The height is growing at that rate since the derivative is positive.)

Problem 3 (20 points)
Evaluate the integrals:

(a) $$\int_0^2 \frac{x^2}{(9-x^3)^{\frac{3}{2}}}\,dx$$

ANSWER: We use a "u-substitution" with $u = 9 - x^3$. Then $du = \frac{du}{dx}dx = (-3x^2)dx$, or $dx = \left(-\frac{1}{3x^2}\right)du$. When $x = 0$, $u = 9 - 0^3 = 9$. When $x = 2$, $u = 9 - 2^3 = 1$. Substituting in the integral we have

$$\int_9^1 \frac{x^2}{u^{\frac{3}{2}}} \times \left(-\frac{1}{3x^2}\right) du = -\frac{1}{3}\int_9^1 u^{-\frac{3}{2}} du = \frac{1}{3}\int_1^9 u^{-\frac{3}{2}} du.$$

Using the power rule we find $-2u^{-\frac{1}{2}} = -2/\sqrt{u}$ as an antiderivative and evaluate the integral as $\frac{1}{3}[(-2/\sqrt{9}) - (-2/\sqrt{1})] = \frac{1}{3}[2 - \frac{2}{3}] = \frac{4}{9}$.

(b) $$\int 2x\sin(x^2)\,dx$$

ANSWER: Again we have to deal with the effect of the chain rule in finding antiderivatives, which we can handle mechanically by a substitution. If we let $u = x^2$, then $du = 2x\,dx$. Hence the integral is just $\int \sin(u)\,du$ which gives $-\cos(u) + C$. Substituting back to express the answer in terms of x we get $-\cos(x^2) + C$.

Problem 4 (18 points)

(a) If $f'(x) = 3x^2 - 4x - \cos(x)$ and $f(0) = 3$, what is the function $f(x)$?

ANSWER: Since f has for its derivative $3x^2 - 4x - \cos(x)$, f is an antiderivative of that function. I.e., f is one of the functions represented by $\int(3x^2 - 4x - \cos(x))\,dx$. Evaluating that we get $x^3 - 2x^2 - \sin(x) + C$. Hence $f(x) = x^3 - 2x^2 - \sin(x) + C$ for some choice of C. But then $f(0) = 0^3 - 2 \times 0^2 - \sin(0) + C = C$, while we have to have $f(0) - 3$. Thus $C = 3$, so $f(x) = x^3 - 2x^2 - \sin(x) + 3$.

(b) (This f has nothing to do with part (a)!)
Let $f(x) = 3x^2 - 2$.

(i) Find a value c in $[0, 2]$ satisfying the Mean Value Theorem for integrals, i.e. $f(c)(2-0) = \int_0^2 f(x)\,dx$.

ANSWER: First we calculate $\int_0^2 (3x^2 - 2)\,dx = \left[x^3 - 2x\right]_0^2 = (8-4) - (0-0) = 4$. Now we know $f(c)(2-0) = 4$, or $f(c) = 2$. But $f(c) = 3c^2 - 2$, so $3c^2 - 2 = 2$, $c^2 = \frac{4}{3}$, $c = \pm\sqrt{\frac{4}{3}}$ or $c = \pm\frac{2}{\sqrt{3}}$. Since $0 < \sqrt{\frac{4}{3}} < 2$ we use the positive square root, so $c = \frac{2}{\sqrt{3}}$ or $c = \frac{2\sqrt{3}}{3}$.

(ii) Find a value c in $[0,2]$ satisfying the Mean Value Theorem for derivatives, i.e. $f'(c) = \dfrac{f(2) - f(0)}{2-0}$.

ANSWER: This time we calculate $f(2) = 3 \times 2^2 - 2 = 10$ and $f(0) = 3 \times 0^2 - 2 = -2$ and get $f'(c) = \frac{10-(-2)}{2-0} = 6$. But $f'(x) = 6x$ in general so $f'(c) = 6c$. Hence $6c = 6$, so $c = 1$. Again that is in the interval $[0,2]$ and we are through.

Problem 5 (18 points)

(a) What is the average value of $3x^2 - 2x + 1$ on the interval $[1,4]$?

ANSWER: We calculate $$\frac{\int_1^4 (3x^2 - 2x + 1)\,dx}{4-1}.$$

The numerator is $\left[x^3 - x^2 + x\right]_1^4 = (64 - 16 + 4) - (1 - 1 + 1) = 52 - 1 = 51$, so the answer is $\frac{51}{3} = 17$.

(b) Find the derivative $G'(x)$ if $G(x) = \int_3^{x^2} \sin(t)\,dt$.

ANSWER: If we wanted $\frac{dG}{d(x^2)}$ we could directly apply the first fundamental theorem of calculus to get $\sin(x^2)$. But $\frac{dG}{dx} = \frac{dG}{d(x^2)} \times \frac{d(x^2)}{dx}$ and $\frac{d(x^2)}{dx} = 2x$, so the answer is $\sin(x^2) \times (2x)$ or $2x\sin(x^2)$.

Problem 6 (14 points)

Use derivatives (differentials) to find approximately the value $\sqrt[3]{7.9}$.

ANSWER: We want the cube root of 7.9. If we let $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$, we are looking for $f(7.9)$. But we "know" $f(8)$, the cube root of 8 is 2. So we use this as the basis for our approximation.

There are several different ways this approximation can be phrased. We could, for example, find the equation $y = mx + b$ for the tangent line to the graph of $f(x)$ at the point $(8,2)$, and then find what value of y on the line corresponds to $x = 7.9$. The more directly computational approach is to say $\Delta y \approx \frac{dy}{dx}\,\Delta x$. Here we have $y = f(x) = x^{\frac{1}{3}}$, so $\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$. Δx is the change in x from the place where we know the function ($x = 8$) to where we want to know it ($x = 7.9$), so $\Delta x = 7.9 - 8 = -0.1$. Δy will be the change in y from the value we know ($y = \sqrt[3]{8} = 2$) to the value we want, so the value we want ($\sqrt[3]{7.9}$) would be $2 + \Delta y$. Putting all of this together: The value of $\frac{dy}{dx}$ at $x = 8$ is $\frac{1}{3}\left(8^{-\frac{2}{3}}\right) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$. Hence $\Delta y \approx \frac{1}{12} \times (-0.1) = -\frac{1}{120}$. Then $\sqrt[3]{7.9} = 2 + \Delta y \approx 2 - \frac{1}{120} = \frac{239}{120}$. (Represented as a decimal fraction that is $1.9916\overline{66}$. That can be compared to the calculator version of $\sqrt[3]{7.9} \approx 1.99163170129$.)