Mathematics 221, Spring 2006 Lecture 3 (Wilson)

First Midterm Exam February 17, 2006 Answers

Problem 1 (18 points)
For each of these functions $f(x)$, find $f'(x)$:

(a) $f(x) = 5x^8 \tan(x)$

Answer: We start with the product rule: We will need each of $5x^8$ and $\tan(x)$ times the derivative of the other. The derivative of $5x^8$ is $5 \times 8 \times x^7 = 40x^7$. The derivative of $\tan(x)$ is $\sec^2(x)$. So the answer is $5x^8 \sec^2(x) + 40x^7 \tan(x)$.

(b) $f(x) = \dfrac{3x^2 - 2x + 1}{x^2 - 5}$

Answer: We have to use the quotient rule. The derivative of $3x^2 - 2x + 1$ is $6x - 2$ and the derivative of $x^2 - 5$ is $2x$, so the answer is $\dfrac{(x^2-5)(6x-2) - 2x(3x^2 - 2x + 1)}{(x^2-5)^2} = \dfrac{2x^2 - 32x + 10}{x^4 - 10x^2 + 25}$.

(c) $f(x) = \sin(\cos(x^2))$

Answer: We use the chain rule: The "outside" function is the sine, whose derivative is the cosine. Moving in one level we have the cosine, whose derivative is −sine. Inside that is x^2, whose derivative is $2x$. Hence the answer is $\cos(\cos(x^2))(-\sin(x^2))(2x) = -2x\sin(x^2)\cos(\cos(x^2))$.

Problem 2 (18 points)
Evaluate the limits, or tell if they don't exist:

(a) $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{2x}$

Answer: We certainly cannot "plug in" $x = 0$ since that would give $\frac{0}{0}$ which is meaningless. We remember that $\displaystyle\lim_{x\to 0}\frac{\sin(x)}{x} = 1$, and this resembles that. So we rearrange things: $\displaystyle\lim_{x\to 0}\frac{\sin(x)}{2x} = \lim_{x\to 0}\frac{1}{2}\times\frac{\sin(x)}{x} = \frac{1}{2}\times\frac{\sin(x)}{x} = \frac{1}{2}\times 1 = \frac{1}{2}$, using the theorem that said we could take a multiplied constant outside the limit.

(b) $\displaystyle\lim_{x\to 3}\frac{x^2}{x^2 - 9}$

Answer: If we think of a value for x "near" 3, on the left side of 3, x^2 will be near 9 but less than 9. Hence the denominator of the fraction will be a small negative number, nearly zero, while the numerator is about 9. The ratio will be very large and negative, getting larger in size as $x \to 0$. Hence the limit from the left is $-\infty$.

If we think of a value for x near 3, but now on the right side of 3, x^2 will be near 9 but greater than 9. Hence the denominator of the fraction will be a small positive number, nearly zero, while the numerator is still about 9. The ratio will be very large and positive, getting larger in size as $x \to 0$. Hence the limit from the left is ∞.

Since the limits from the two sides don't agree, the two-sided limit given in the problem DOES NOT EXIST.

(c) $\lim_{x \to 3} \frac{x^2}{(x-9)^2}$

Answer: Unlike in (b), this time as $x \to 3$ the denominator does not go to zero. Hence we have a rational function where the denominator is not going to zero, and a theorem tells us that is continuous. So we this time we can "plug in" $x = 3$. That gives us as the answer $\frac{3^2}{(3-9)^2} = \frac{9}{36} = \frac{1}{4}$.

Comment: I had intended this problem to read $\lim_{x \to 3} \frac{x^2}{(x-3)^2}$ and mistyped it. I did not want to change it at the exam time, that always causes confusion and this did not seem worth it. But as an example, for that version of the problem: The reasoning in (b) almost applies, but this time the denominator (being a square) is positive on either side of 3 and so the limit from either side is positive ∞. Hence the two-sided limit does exist and is ∞.

Problem 3 (15 points)
For the function $f(x) = 3x^2 - 4x + 1$:

(a) What is the average rate of change of $f(x)$ as x goes from 0 to 2?

Answer: We divide the change in f from $x = 0$ to $x = 2$ by the change in x. We get $\frac{f(2) - f(0)}{2 - 0} = \frac{5 - 1}{2} = \frac{4}{2} = 2$.

(b) What is the derivative of $f(x)$ as a function?

Answer: We can use the power rule and other simple rules to get $f'(x) = 6x - 4$.

(c) What is the instantaneous rate of change of $f(x)$ at $x = 0$?

Answer: We evaluate $f'(0)$ and get $6 \times 0 - 4 = -4$.

(d) What is the instantaneous rate of change of $f(x)$ at $x = 1$?

Answer: Similarly, $f'(1) = 6 \times 1 - 4 = 2$.

(e) What is the instantaneous rate of change of $f(x)$ at $x = 2$?

Answer: And $f'(2) = 6 \times 2 - 4 = 8$.

Comment: Note that the instantaneous rate of change in the middle of the interval $[0, 2]$ is the same as the average rate of change over the whole interval. This does not always happen, but it would always happen if f is given by a polynomial of degreee 2 or less.

Problem 4 (15 points)
Use the definition of the derivative as a limit to find $f'(x)$ for $f(x) = 2x^2 + x + 3$.
Answer:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{2(x+h)^2 + (x+h) + 3 - 2x^2 - x - 3}{h} \\ &= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 + x + h + 3 - 2x^2 - x - 3}{h} \\ &= \lim_{h \to 0} \frac{4xh + 2h^2 + h}{h} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0}(4x+2h+1) \\
&= 4x+1.
\end{aligned}
$$

Problem 5 (16 points)
For $f(x) = \sin(2x) + 3$:

(a) Find an equation for the tangent line to the graph of $f(x)$, at the point $(0, 3)$ on the graph.

Answer: We need to find an equation in a form such as $y = mx + b$ where m is the slope. So first we find the slope. That will be $f'(0)$ so we find $f'(x)$ and evaluate it at $x = 0$. Using the chain rule we get $f'(x) = 2\cos(2x)$. Then $f'(0) = 2\cos(0) = 2 \times 1 = 2$. So now we know the slope, $m = 2$. We can proceed in at least two different ways. One, the most general, is to say the line goes through the point $(x_0, y_0) = (0, 3)$ so the equation can be written as $y - y_0 = m(x - x_0)$. But in this case we note that the point $(0, 3)$ is on the y-axis, i.e. 3 is the y-intercept, so b must be 3. Hence the equation is $y = 2x + 3$.

(b) What is the second derivative $f''(x)$?

Answer: We already have $f'(x) = 2\cos(2x)$, so differentiating again we get (again using the chain rule) $f''(x) = -4\sin(2x)$.

(c) What is the second derivative $f''(\frac{\pi}{4})$?

Answer: We set $x = \frac{\pi}{4}$ in $f''(x) = -4\sin(2x)$. Then $2x = \frac{\pi}{2}$, and $\sin(\frac{\pi}{2}) = 1$, so $f''(\frac{\pi}{4}) = -4\times 1 = -4$.

Problem 6 (18 points)

(a) For $\dfrac{3x-1}{x+3}$, what is $\lim_{x\to 1} f(x)$?

Give reasons for your answer, i.e. don't just give a number. Your reasons should relate to theorems we have studied: You do not need to cite theorems by number, but you do need to tell in some form what they say.

Answer: The fraction $\dfrac{3x-1}{x+3}$ describes a rational function, a ratio of two polynomials. We have a theorem (Theorem A in section 2.9) that tells us that a rational function is continuous anywhere its denominator is not zero. In this case the denominator $x + 3$ takes the value 4 at $x = 1$, certainly not zero, so the function is continuous at $x = 1$. The definition of continuity tells us that the limit as $x \to 1$ will be the same as the value of the function at $x = 1$. Hence $\lim_{x\to 1} \dfrac{3x-1}{x+3} = \dfrac{3\times 1-1}{1+3} = \dfrac{2}{4} = \dfrac{1}{2}$.

(b) For $f(x) = x^2 - 2$: Show that there must be some value of x for which $f(x) = 0$, i.e. "there is a square root of 2".

Don't just give some numeric approximation to $\sqrt{2}$. Use some theorem(s) we have studied.

Answer: The main theorem we have that says a function must take some particular value is the Intermediate Value Theorem. Since $f(x)$ is given by a polynomial, it is continuous at any value of x. So if we can find some x that makes $f(x)$ negative and some other x that makes $f(x)$ positive, the IVT will guarantee there is an x between those two values that makes $f(x)$ zero and we will be through. If I try for example $x = 0$, $f(0) = 0 - 2$ is negative. And if I try $x = 2$ as another example, $f(x) = 4 - 2$ is positive. Hence we know not only that $f(x)$ is zero for some x but even a little more, that there is some x between 0 and 2 making $f(x) = 0$.