Mathematics 221, Fall 2007 **Lecture 1 (Wilson)**

Final Exam December 17, 2007 ANSWERS

Problem 1 (30 points)

Find the derivative $\frac{dy}{dx}$ for:

(a) $y = \int_1^{x^2} \sin(3t^2 - 2)\, dt$

Answer: If we wanted $\frac{dy}{dx^2}$ we could just use the Fundamental Theorem to get $\sin(3(x^2)^2 - 2) = \sin(3x^4 - 2)$. But since we want $\frac{dy}{dx}$ we use the chain rule, $\frac{dy}{dx} = \frac{dy}{dx^2} \times \frac{dx^2}{dx}$ $= \sin(3x^4 - 2) \times 2x$ which we could rewrite as $2x \sin(3x^4 - 2)$.

(b) $y = \sin^{-1}(3x^2)$

Answer: The derivative of $\sin^{-1}(x)$ would be $\frac{1}{\sqrt{1-x^2}}$ and we then change the variable and use the chain rule to get $6x \times \frac{1}{\sqrt{1-(3x^2)^2}}$ or $\frac{6x}{\sqrt{1-9x^4}}$.

(c) $y = \left(e^{x^2}\right) \ln(x)$

Answer: We start with the product rule, $\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^2}\right) \ln(x) + e^{x^2} \frac{d}{dx} \ln(x)$. Then $\frac{d}{dx}\left(e^{x^2}\right) = e^{x^2} \frac{d}{dx} x^2 = 2x\, e^{x^2}$, and $\frac{d}{dx} \ln(x) = \frac{1}{x}$. Putting it all together, $\frac{dy}{dx} = 2xe^{x^2} \ln(x) + \frac{e^{x^2}}{x}$.

Problem 2 (25 points)

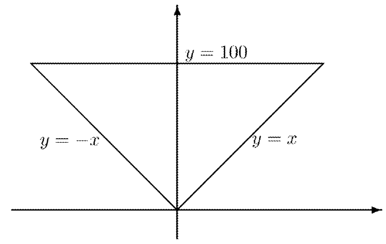
Find all maximum and minimum points for the function $f(x) = x^3 - 3x^2 - 9x + 2$ on the closed interval $-4 \le x \le 3$.

Answer: We need to consider critical points as well as the end points of the interval. The derivative $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3)$ can be factored as $3(x-3)(x+1)$, so we see it gives zero at $x = 3$ and at $x = -1$, or you could use the quadratic formula or other means to find those roots. The derivative does not fail to exist at any x. So the critical points correspond to $x = -1$ and $x = 3$. The second derivative $f''(x) = 6x - 6$: At $x = -1$ that gives $f''(-1) = -12 < 0$, and at $x = 3$ we have $f''(3) = 12 > 0$. So the graph is concave downward and there is a local maximum at $x = -1$, concave upward with a local minimum at $x = 3$. Looking at the interval $-4 \le x \le 3$ we see that $x = -1$ is safely inside and there is a local maximum there, where $f(x) = f(-1) = (-1)^3 - 3 \times (-1)^2 - 9 \times (-1) + 2 = 7$. But $x = 3$ is both a critical point for the function in general and an endpoint of the interval! That does not hurt anything, we have a local minimum at $x = 3$ with $f(x) = f(3) = 3^3 - 3 \times 3^2 - 9 \times 3 + 2 = -25$. We still have the remaining endpoint, $x = -4$, where $f(x) = f(-4) = (-4)^3 - 3 \times (-4)^2 - 9 \times (-4) + 2 = -74$. At $x = -4$ we have the derivative $f'(-4) = 3 \times (-4)^2 - 6 \times (-4) - 9 = 63$ which is positive, so the function was increasing as we came to the interval from the left, so there is a local minimum there.

Comparing values: At $x = -4$, $f(x) = -74$; at $x = -1$, $f(x) = 7$; at $x = 3$, $f(x) = -25$. Hence the overall, absolute, or global maximum is at $x = -1$ with $f(x) = 7$, and the absolute minimum is at $x = -4$ with $f(x) = -74$, while $x = 3$ with $f(x) = -25$ gives a local minimum but not an absolute maximum.

Problem 3 (25 points)

A dam in the form of an isoceles right triangle has for its horizontal top edge the hypotenuse of the triangle. It stands 100 feet high. You can picture it in relation to a set of axes in the diagram to the right.



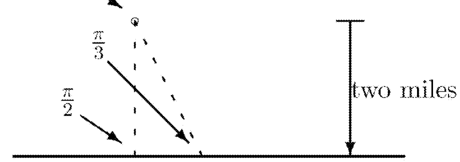
Water weighing 62.4 pounds per cubic foot stands against one side of the dam from the bottom all the way to the top. That produces a pressure against the dam at a distance h feet down from the top that is $62.4\,h$ pounds per square foot, 0 at the top and 6240 pounds per square foot at the bottom point (where $y = 0$ for the axes in the picture).

Set up but do not evaluate an integral that would compute the total force agains the dam resulting from the water.

Answer: We think of the dam cut into horizontal strips of height Δy, from a strip at the bottom where $y = 0$ up to the top where $y = 100$. A strip at height y extends from $x = -y$ at the left to $x = y$ at the right, so horizontally it measures $y - (-y) = 2y$ feet, and vertically it measures Δy, so it has area $2y\,\Delta y$. At height y how far is it below the surface of the water? The depth increases as y goes down, and it is 0 when $y = 100$, so the depth at height y is $100 - y$ feet and the pressure at that point is $62.4 \times (100 - y)$ pounds per square foot. Putting the area and the pressure together we get $62.4 \times (100 - y) \times 2y \times \Delta y$ as the force on the strip. Adding the forces on many strips and taking a limit as the strips are narrower but there are more and more of them, we get $\int_0^{100} 62.4(100 - y)(2y)dy$ as an integral to compute the total force. We can make that a little prettier as $62.4 \int_0^{100} (200y - 2y^2)dy$.

Problem 4 (24 points)

A lighthouse 2 miles from a straight shoreline rotates counter-clockwise (viewed from above) once per minute. The lighthouse sends out a beam of light, which is thus rotating at 2π radians per minute. The picture at the right shows the situation at the two instants that matter in the questions below.



(a) At an instant when the light beam is aimed straight toward the shoreline, so it hits the shore at a right angle, how fast in miles/minute is the spot where the beam hits the shore moving?

Answer: It will help to introduce some names for variables: Let $x(t)$ be distance horizontally (positive to the right) along the shoreline from the point where the light beam hits the shoreline at the closest point, the one marked $\frac{\pi}{2}$ in the picture, to where the beam hits the shore at time t. Let $\theta(t)$ be the angle between the beam and some fixed line, let's make it the vertical line in the picture going through $x = 0$, at time t. Then what we are after is the value of $\frac{dx}{dt}$ when $\theta = 0$. We know that $\frac{d\theta}{dt} = 2\pi$ in radians per minute. Using the chain rule, $\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt}$. So we need to find $\frac{dx}{d\theta}$. If we look at the triangle with two dashed sides in the picture; θ is the angle at the top, x is the length of the side along the shore, and the vertical side is 2 miles long, so $\frac{x}{2} = \tan\theta$, i.e.$x = 2\tan\theta$. Then $\frac{dx}{d\theta} = 2\sec^2\theta$. Putting the pieces together, at any time t, $\frac{dx}{dt} = 2\sec^2(\theta) \times 2\pi$ if we just substitute in the value of θ at that time. When the light beam is at right angles to the shoreline, $\theta = 0$ (or some multiple of 2π): $\cos 0 = 1$ so $\sec 0 = \frac{1}{1} = 1$, so $\frac{dx}{dt} = 2 \times 1^2 \times 2\pi = 4\pi$ miles per minute.

(b) Slightly later, when the beam is at $\frac{\pi}{3}$ radians to the shore rather than $\frac{\pi}{2}$, how fast is the lighted spot moving?

Answer: We use the same formula as above for $\frac{dx}{dt}$, but what is the value of θ at this time? Looking at the triangle in the picture, where one angle is $\frac{\pi}{3}$, θ is the other non-right-angle in the triangle so it is $\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$ radians. Hence $\cos\theta = \frac{\sqrt{3}}{2}$ and $\sec\theta = \frac{2}{\sqrt{3}}$, so $\frac{dx}{dt} = 2 \times \left(\frac{2}{\sqrt{3}}\right)^2 \times 2\pi$ which gives $\frac{16}{3}\pi$ miles per minute.

(As somewhat of a check: Note that $\frac{16}{3}$ is bigger than 5 and hence bigger than 4, and as the beam moves out along the shore it ought to be speeding up, so this looks good.)

Problem 5 (30 points)

Evaluate the integrals. (Notice that some are definite and some indefinite.)

(a) $\int \sin^5\left(\frac{x}{3}\right)\cos\left(\frac{x}{3}\right)\,dx$

Answer: Since the derivative of the sine is the cosine, this is all set up to use the substitution $u = \sin\left(\frac{x}{3}\right)$. Then $du = \frac{1}{3}\cos\left(\frac{x}{3}\right)\,dx$ which is $\frac{1}{3}$ of the rest of what is to be integrated. So $\int \sin^5\left(\frac{x}{3}\right)\cos\left(\frac{x}{3}\right)\,dx = 3\int u^5\,du = 3 \times \frac{u^6}{6} + C$. Simplifying the $\frac{3}{6}$ and substituting back the original variable we get $\frac{1}{2}\sin^6\left(\frac{x}{3}\right) + C$.

(b) $\int_0^1 \left(\sqrt{t^5 + 2t}\right)(5t^4 + 2)\,dt$

Answer: The portion $5t^4 + 2$ is exactly the derivative of what is inside the square root. So this works easily with the substitution $u = t^5 + 2t$. But we need to be careful about the limits on the integral. Using that substitution, $du = (5t^4 + 2)dt$ so the integral becomes $\int_?^? u^{\frac{1}{2}}du$, but what are the limits in terms of u? When $t = 0$, $u = 0^5 + 2 \times 0 = 0$, and when $t = 1$, $u = 1^5 + 2 \times 1 = 3$. So the integral becomes $\int_0^3 u^{\frac{1}{2}}du = \frac{2}{3}u^{\frac{3}{2}}\Big]_0^3 = \frac{2}{3} \times 3^{\frac{3}{2}} = \frac{2}{3}\sqrt{27}$. A simpler way you could write it is $2\sqrt{3}$.

(c) $\int \frac{dx}{2 + (x-1)^2}$

Answer: This looks a lot like the derivative of the arctangent. Let's start by making the 2 into a 1: We move $\frac{1}{2}$ outside the integral and have $\frac{1}{2}\int \frac{dx}{1 + \frac{(x-1)^2}{2}}$. Now it would be nice to have $u^2 = \frac{(x-1)^2}{2}$ so we substitute $u = \frac{x-1}{\sqrt{2}}$, with $du = \frac{1}{\sqrt{2}}dx$ so $dx = \sqrt{2}du$.

Now the integral looks like $\frac{\sqrt{2}}{2}\int \frac{du}{1+u^2}$ which we recognize as exactly the arctangent, so in terms of u the answer is $\frac{\sqrt{2}}{2}\tan^{-1}(u) + C$. Putting back the original variable we have $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C$.

Problem 6 (20 points)

Suppose x and y are quantities varying with time t, and that at all times they are related by $x = \sin(y)$.

(Note that in that case, at any time t, we would have $-1 \le x \le 1$.)

Suppose we also know that at any time, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

(a) Write a formula giving y as a function of x.

Answer: Since $x = \sin(y)$, y should be $\sin^{-1}(x)$ (or $\arcsin(x)$ to use the other notation, if the range and domain are right. But the restrictions noted, $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, are exactly what are needed to match the definition of the inverse sine so that is the answer.

(b) What is $\frac{dy}{dx}$? We could use the theorem about derivatives of inverses of functions, but now that we have a formula for the derivative of $\sin^{-1}(x)$ it is probably simpler just to write down $\frac{1}{\sqrt{1-x^2}}$.

Answer:

(c) If $\frac{dx}{dt} = 3$ and $x = \frac{\sqrt{2}}{2}$ and $y = \frac{\pi}{4}$, what is $\frac{dy}{dt}$?

Answer: The chain rule tells us $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$. So $\frac{dy}{dt} = \frac{1}{\sqrt{1-x^2}}\frac{dx}{dt}$. Since we are given that $\frac{dx}{dt} = 3$ and $x = \frac{\sqrt{2}}{2}$ we can put those in and get $\frac{dy}{dt} = \frac{1}{\sqrt{1-\frac{1}{2}}} \times 3 = 3\sqrt{2}$.

Problem 7 (26 points)

(a) Find the total area between the graph of $f(x) = x^2 - 2x$ and the x-axis, for $1 \le x \le 3$.

Answer: At some values of x this graph lies below the x-axis (e.g. at $x = 1$ where $f(x) = f(1) = 1^2 - 2 \times 1 = -1$) while at other values of x it is above the x axis. So we need to divide the interval $[1, 3]$ into pieces where f is positive on some and negative on the others. You can plot the graph, or just work algebraically. If we write $f(x) = x^2 - 2x = x(x-2)$ we see: f is zero if $x = 0$ (not in the interval) or $x = 2$. Since it is continuous, it can change from negative to positive or vice-versa only where it is zero. Evaluating $f(1) = -1$ (done above) we see f must be negative on the entire interval $0 < x < 2$, or for our area calculation the negative part is $1 < x < 2$. Since $f(3) = 9 - 6 = 3 > 0$, f is positive to the right of $x = 2$. Hence we break the interval $[1, 3]$ into two pieces, $[1, 2]$ where f is negative and $[2, 3]$ where it is positive. So the area is $-\int_1^2 (x^2 - 2x)dx + \int_2^3 (x^2 - 2x)dx$. The indefinite integral $\int (x^2 - 2x)dx = \frac{x^3}{3} - x^2 + C$ so the first part of the area is $-\left(\left[\frac{x^3}{3} - x^2\right]_1^2\right)$ $= -\left[\left(\frac{8}{3} - 4\right) - \left(\frac{1}{3} - 1\right)\right] = -\left[-\frac{4}{3} + \frac{2}{3}\right] = \frac{2}{3}$. The second integral, from the region where f is positive, gives $\left(\frac{27}{3} - 9\right) - \left(\frac{8}{3} - 4\right) = \frac{4}{3}$. The total area, the area below the axis added to the area above, is $\frac{2}{3} + \frac{4}{3} = 2$ square units.

(b) A function $g(x)$ has its derivative proportional to the function itself, i.e. $g'(x) = k\,g(x)$ for some constant k. Find a formula for $g(x)$ if $g(0) = 3$ and $g'(0) = 6$.

Answer: We know if $g'(x) = k\,g(x)$ then $g(x) = Ce^{kx}$ for some constant C. So we just need to find values for C and k that make this formula fit the given data. $g(0) = Ce^0 = C \times 1 = C$, and we are given $g(0) = 3$, so C must be 3. Taking the derivative, $g'(x) = kCe^{kx} = 3ke^{kx}$, so $g'(0) = 3ke^0 = 3k \times 1$, but we know $g'(0) = 6$. So $3k = 6$, hence $k = 2$, and we now know $g(x) = 3e^{2x}$.

Problem 8 (20 points)

(a) Find an equation for the tangent line to the graph of $y = \cosh(x)$ at $(\ln 2, \frac{5}{4})$.

Answer: We take the derivative, $y' = \sinh(x)$ (you could do this using instead the representation $\cosh(x) = \frac{e^x+e^{-x}}{2}$) and find the slope of the tangent line is $\sinh(\ln 2) = \frac{e^{\ln 2}-e^{-\ln 2}}{2} = \frac{2-\frac{1}{2}}{2} = \frac{3}{4}$. Now we just write the equation for the line through $(\ln 2, \frac{5}{4})$ with slope $\frac{3}{4}$: $y - \frac{5}{4} = \frac{3}{4}(x - \ln 2)$. You can rearrange this in various ways, e.g. $y = \frac{3}{4}x - \frac{3}{4}\ln 2 + \frac{5}{4}$.

(b) Assume $\ln 2 = 0.69$, which is approximately right. Use the linearization of $\cosh(x)$ at $x = \ln 2$ to give an approximation to $\cosh(0.68)$.

(Make use of the fact that $0.68 = 0.69 - 0.01$.)

Answer: We can write out the formula from §3.8 for the linearization $L(x)$, which amounts to the equation above for the tangent line. We have $L(x) = \frac{3}{4}(x - \ln 2) + \frac{5}{4}$. Using $x = 0.68$ and $\ln 2 \approx 0.69$, that becomes $\frac{3}{4}(0.68-0.69)+\frac{5}{4} = 0.75\times(-0.01)+1.25 = -0.0075 + 1.25 = 1.2425$.

(The actual value of $\cosh(0.68)$ is more like 1.240247362, so this is not too bad!)