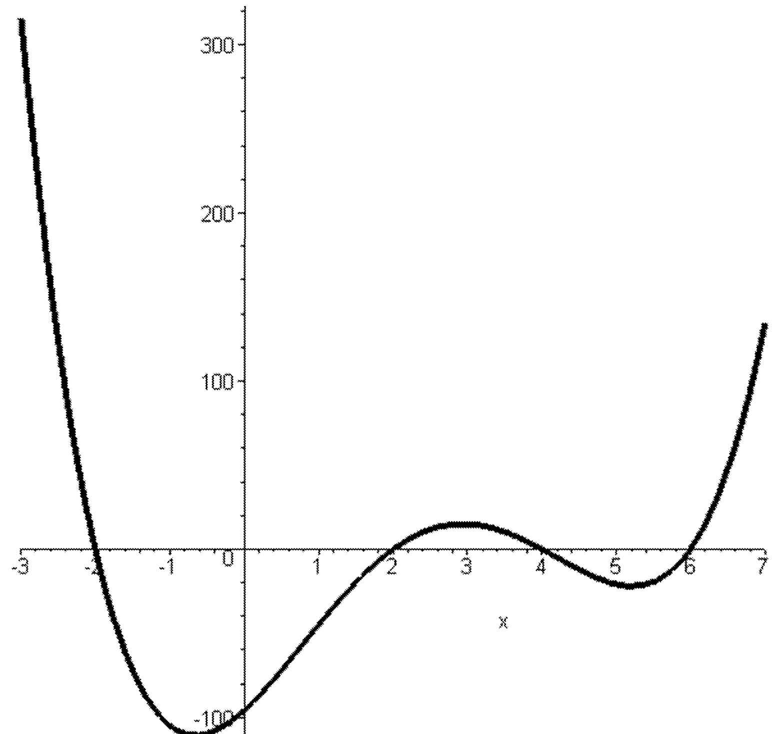
Mathematics 221, Fall 2007 Lecture 1 (Wilson)

Second Midterm Exam November 29, 2007 ANSWERS

<u>Problem 1</u> (12 points)

At the right is the graph of a function $f(x)$, defined only for $-3 \le x \le 7$. Use that graph in answering the following.
Your answers may not be exact, reading from the graph, but come as close as you can.



(a) For what value(s) of x is $f(x) = 0$?

<u>Answer:</u> We see the graph crosses the x-axis at approximately $x = -2$, $x = 2$, $x = 4$, and $x = 6$.

(b) For what value(s) of x is $f''(x) = 0$?

<u>Answer:</u> The second derivative changes sign, the graph changes between concave up and down, at about $x = 0.9$ and $x = 4$, so if it is continuous f'' must be zero there.

(c) On what interval(s) $[a, b]$ is $f(x)$ decreasing?

<u>Answer:</u> The graph is sloping down on $[-3, -0.8]$ and $[3, 5.1]$.

(d) On what interval(s) $[a, b]$ is the graph concave upward?

<u>Answer:</u> Using what we had in (b), the graph is concave upward on $[-3, 0.9]$ and $[4, 7]$.

(e) At what value(s) x does $f(x)$ have a critical point?

<u>Answer:</u> Our textbook defines a critical point to be a place where $f' = 0$ or f' does not exist. This seems to be a smooth curve such that $f'(x)$ exists at every point in the interior of the domain. The places where $f'(x) = 0$, where the graph levels off, are about $x = -0.8$, $x = 3$, and $x = 5.1$.

Some people define critical points to include the endpoints of the interval $x = -3$ and $x = 7$, also.

(f) At what value(s) x does $f(x)$ have an inflection point?

<u>Answer:</u> This is almost the same as (b): It could happen that $f''(x) = 0$ but f'' does not change sign a x, but in the picture that does not happen. So the x values are $x = 0.9$ and $x = 4$.

Problem 2 (12 points)
Evaluate the integrals:
(Express your answers in terms of the original variable x.)

(a) $\int x \sqrt[4]{1-x^2}\, dx$.

Answer: If we let $u = 1 - x^2$, then $\sqrt[4]{1-x^2}$ becomes a simple $u^{\frac{1}{4}}$, and the x in front of the square root lets us construct du: $du = -2x\, dx$. So the integral becomes $\int u^{\frac{1}{4}} \frac{-du}{2}$ or $-\frac{1}{2} \int u^{\frac{1}{4}}\, du$. Then we get $-\frac{1}{2}\left(\frac{1}{5/4}\right) u^{\frac{5}{4}} + C = -\frac{2}{5} u^{\frac{5}{4}} + C$. Substituting back to the original variable we get $-\frac{2}{5}(1-x^2)^{\frac{5}{4}} + C$.

(b) $\int (\sin(2x))^3 \cos(2x)\, dx$.

Answer: If we let $u = \sin(2x)$ then the $\cos(2x)$ factor will help create du. Letting $u = \sin(2x)$ we have $du = 2\cos(2x)\, dx$ (don't forget the chain rule!), so the integral can be rewritten as $\int u^3 \frac{du}{2} = \frac{1}{2}\frac{u^4}{4} + C$ $= \frac{1}{8}(\sin(2x))^4 + C$.

(c) $\int \cos^2 x\, dx$

Answer: The book did examples like this using the trig identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$. The integral becomes $\int \frac{1}{2}(1 + \cos 2x)\, dx = \frac{1}{2}\left(\int dx + \int \cos 2x\, dx\right)$. You can use a substitution like $u = 2x$ to work out that last part if you need it, and the answer is $\frac{x}{2} + \frac{1}{4}\sin 2x + C$.

Problem 3 (14 points)

Part I Let $f(x) = 2x^3 + 3x^2 - 12x + 4$ (for all numbers x).

(a) Does $f(x)$ have any absolute (global) maxima or minima? (Yes/No) Give a reason for your answer!

Answer: The function f is defined for all numbers. As we go to the left, negative x, it takes on ever greater negative values, with no bound, so there is no absolute minimum. Similarly as we go to the right the function increases without bound so there is no absolute maximum.

(b) Find all maxima and minima of f: Identify each as to maximum or minimum and local or absolute, and give reasons for your answers.

Answer: We get $f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x+2)(x-1)$, so $f'(x) = 0$ for $x = -2$ and $x = 1$. As a polynomial, f is differentiable anywhere. So the only critical points occur when $x = -2$ and $x = 1$. If we differentiate again, $f''(x) = 6(2x+1)$. Thus $f''(-2) = 6(-4+1) < 0$, and $f''(1) = 6(2+1) > 0$. Hence f has a local maximum at $x = -2$ and a local minimum at $x = 1$ by the second derivative test.

Part II Using the same formula $f(x) = 2x^3 + 3x^2 - 12x + 4$ but this time restricting the domain so that f is only defined for $-1 \le x \le 2$:

(c) Does $f(x)$ have any absolute maxima or minima? (Yes/No) Give a reason for your answer!

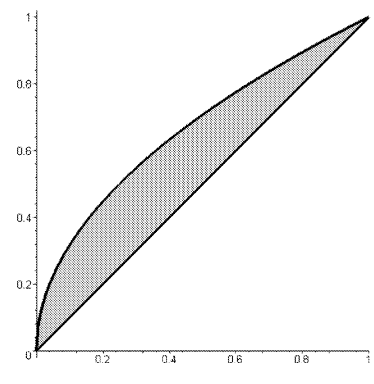
Answer: It must! A continuous function on a closed interval always has an absolute maximum and an absolute minimum on that interval.

(d) Find all maxima and minima of f: Identify each as to maximum or minimum and local or absolute, and give reasons for your answers.

<u>Answer:</u> We have already found the potential critical points above, at $x = -2$ and $x = 1$. Now we have to include endpoints $x = -1$ and $x = 2$, and filter those critical points to leave out any that are not in the interval $[-1, 2]$. We are left with the values $x = -1$, $x = 1$, and $x = 2$ to consider. Computing $f(x)$ at each of those values we get $f(-1) = 17$, $f(1) = -3$, and $f(2) = 8$. Comparing these we can see that the absolute maximum value for f is 17, taken when $x = -1$, and the absolute minimum value is -3, taken when $x = 1$. At the point $(2, 8)$ the derivative $f'(2) = 24 > 0$, so the function is climbing toward that right-hand endpoint, so this gives a local maximum.

<u>Problem 4</u> (12 points)

The region between $y = x$ and $y = \sqrt{x}$ (for $0 \leq x \leq 1$) is shown at the right. This region is rotated about the y-axis to produce a solid which is conical on the outside and has a flared hole in the inside.



Set up and evaluate an integral to compute the volume of this solid.
You are free to choose whether to use "washers" or "shells".
I will do this both ways:
Using washers, we want to partition (slice across) the axis of rotation, the y-axis. A representative slice at height y, for $0 < y < 1$, cuts an approximately rectangular piece from the region. The height is Δy, and the rectangle extends from $x = y^2$ to $x = y$ horizontally. The washer created by rotating that rectangle around the y-axis has thickness Δy. Its outer rim is a circle with radius y and the inner circle has radius y^2, so the area of a face of the washer is $\pi y^2 - \pi y^4$ and the volume contributed is $\pi(y^2 - y^4)\Delta y$. Adding those volumes, for $0 \leq y \leq 1$, and then taking a limit for finer partitions, we get $\pi \int_0^1 \left(y^2 - y^4)dy\right)$. Evaluating that integral we get $\pi \left(\left[\frac{y^3}{3} - \frac{y^5}{5} \right]_0^1 \right) = \frac{2}{15}\pi$.
Now, redoing the problem using shells: This time we slice parallel to the axis of rotation, so we are slicing across the x-axis. A piece of the x-axis of length Δx, at x, corresponds to a vertical slice from the region: It has left-to-right width Δx, and extends from the lower "curve" $y = x$ up to the curve $y = \sqrt{x}$, so its length in that vertical direction is $\sqrt{x} - x$. When the rectangle is rotated around the y-axis it sweeps out a "shell": The radius of the shell is x. The height from bottom to top is $\sqrt{x} - x$. The thickness is Δx. If we cut the shell and roll it out flat we get, approximately, a rectangular slab with length $2\pi x$, width $\sqrt{x} - x$, and thickness Δx. Hence its volume is $2\pi x(\sqrt{x} - x)\Delta x$. Adding up and letting the slices get thinner we come to the integral $2\pi \int_0^1 x(\sqrt{x} - x)dx = 2\pi \int_0^1 (x^{\frac{3}{2}} - x^2)dx$. Evaluating we get (thankfully!) the same number, $\frac{2\pi}{15}$.

Problem 5 (12 points)

(a) If $f(x) = 4x^2 - 3x + 1$, $a = 1$, and $b = 3$, find a number that works as c in the Mean Value Theorem for Derivatives, i.e. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Show how you found c, and work out both $f'(c)$ and $\frac{f(b)-f(a)}{b-a}$ to verify that your choice of c is correct.

Answer: First we calculate $f'(x) = 8x-3$. Next we want $\frac{f(b)-f(a)}{b-a}$: $f(b) = f(3) = 4\times 9-3\times 3+1 = 28$ and $f(a) = f(1) = 4 \times 1 - 3 \times 1 + 1 = 2$, so $\frac{f(b)-f(a)}{b-a} = \frac{28-2}{3-1} = 13$. Now we know we need to find a number c between 1 and 3 such that $f'(c) = 8c - 3$ is equal to 13. Solving, $8c - 3 = 13$, $8c = 16$, and $c = \frac{16}{8} = 2$ which is indeed between 1 and 3.

(b) Evaluate $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x^3}$.

Be sure to show why rules or theorems apply where you use them.

Answer: We would like to use L'Hôpital's rule. We see that the limit (at 0) of the numerator is $0 - \sin 0 = 0$ and the limit of the denominator is $0^3 = 0$, and that both the numerator and denominator separately are differentiable functions, justifying the use of L'Hôpital's rule, so we can substitute the limit of the ratio of the derivatives: $\lim\limits_{x\to 0} \dfrac{1 - \cos x}{3x^2}$ would be the same as the original limit. But again we see this has the form $\frac{0}{0}$, so again we check that the numerator and denominator are differentiable functions and then take the derivative of numerator and denominator to use L'Hôpital's rule, getting $\lim\limits_{x\to 0} \dfrac{\sin x}{6x}$. But this again goes to $\frac{0}{0}$! So we again note the numerator and denominator are differentiable functions, we apply the rule again, and differentiating we want to evaluate $\lim\limits_{x\to 0} \dfrac{\cos x}{6}$. Finally! As $x \to 0$ this gives us $\frac{1}{6}$ as the answer.

Problem 6 (15 points)

(a) Evaluate $\displaystyle\int_0^1 \frac{x+1}{(x^2+2x+2)^3} dx$.

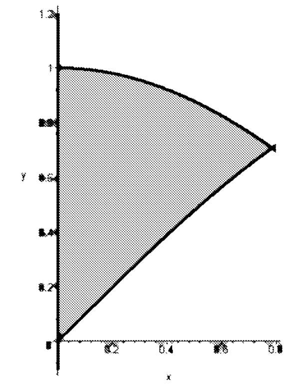
Answer: We note that the derivative of $x^2 + 2x + 2$, $2x + 2$, is a constant multiple of the $x + 1$ we see in the numerator, so we try letting $u = x^2 + 2x + 2$. Then $du = (2x + 2)dx$ so the integral can be rewritten as $\displaystyle\int_?^? \frac{\frac{1}{2}du}{u^3}$ where we still have to work out the limits on the integral: When $x = 0$, $u = 0^2 + 2 \times 0 + 2 = 2$, and when $x = 1$, $u = 1^2 + 2 \times 1 + 2 = 5$, so we evaluate $\frac{1}{2}\displaystyle\int_2^5 u^{-3} du$. That gives us $\left(\frac{1}{2}\right)\left(\frac{1}{-2}\right) u^{-2}\Big]_2^5 = -\frac{1}{4}\left(\frac{1}{25} - \frac{1}{4}\right) = \frac{1}{4}\left(\frac{25}{100} - \frac{4}{100}\right) = \frac{21}{400}$.

(b) Evaluate $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan x \sec^2 x \, dx$.

Answer: The quick way to do this problem is to note that the integrand, $\tan x \sec^2 x$, is an odd function: Subsituting $-x$ for x changes only the sign on $\tan x$ and does not change $\sec x$ or $\sec^2 x$ at all. Hence the integral from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$ must be zero and we are through.

You can also work this out directly: Since the derivative of $\tan x$ is $\sec^2 x$, the substitution $u = \tan x$ changes the integral to $\displaystyle\int_{-1}^1 u \, du = \frac{u^2}{2}\Big]_{-1}^1 = \frac{1}{2} - \frac{1}{2} = 0$.

(c) Set up and evaluate an integral to find the area of the "curved triangle" to the right of the y-axis, below $y = \cos x$, and above $y = \sin x$.



Answer: If we take a vertical slice, across the x-axis, at x and with width Δx, its lower end is on the curve $y = \sin x$ and its upper end is on $y = \cos x$. Hence its height is $\cos x - \sin x$ and with width Δx its area is $(\cos x - \sin x)\Delta x$ and we can see we will want an integral of the form $\int_?^? (\cos x - \sin x)dx$. But what will be the limits of integration? The left edge of the region is the y-axis, where $x = 0$, so that is the lower limit. The upper limit has to be where the curves cross, i.e. the first positive number where $\cos x = \sin x$. You might just recall that as $x = \frac{\pi}{4}$. But if you did not, you could picture the unit circle, where the coordinates of the point at angle θ measured around from the positive x-axis are $(\cos\theta, \sin\theta)$. Hence the place where $\cos\theta = \sin\theta$ is where the line $y = x$ meets the unit circle, half way up to a right angle, hence at $\frac{\pi}{4}$. So we need to evaluate $\int_0^{\frac{\pi}{4}} (\cos x - \sin x)dx = [\sin x + \cos x]_0^{\frac{\pi}{4}}$ $= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) = \sqrt{2} - 1$.

Problem 7 (11 points)

A rectangular box with a square base needs to be painted, but the bottom won't be painted. (So there are four sides plus the top that need to be painted.) The box needs to contain 500 cubic inches.

What should be the dimensions of the box in order to minimize the area that has to be painted? Be sure to give both the height of the box and the length of a side of the base.

Show your work. Explain why you know your answer minimizes the area rather than maximizing it!

Answer: Let s be the length of a side of the square base. Let h be the height of the box. Then the volume of the box is $h\,s^2$. The area of the top is s^2 and the area of each of the four vertical sides is $h\,s$, so the total area to be painted is $s^2 + 4hs$. We want to minimize that area, but it involves two variables h and s. But we are also told the volume is to be 500 cubic inches, so $h\,s^2 = 500$ if we make all our linear measurements in inches. We can rewrite that as $h = \frac{500}{s^2}$, and then subsitute for h in the area formula, to get that the painted area is $s^2 + 4 \times \frac{500}{s^2} \times s$ or $A(s) = s^2 + \frac{2000}{s}$. We want to find the s that minimizes this that expression. We look for critical points and endpoints: The least value for s that makes sense is $s = 0$. If s gets large, $h = \frac{500}{s^2}$ gets small, but there is no evident largest possible value for s. However we see that the area $A(s) = s^2 + \frac{2000}{s}$ would grow arbitrarily large as $s \to \infty$, so a right endpoint would not be the place to look for a minimum anyway. The derivative of the area function, $A'(s) = 2s - \frac{2000}{s^2}$, will exist for any s other than $s = 0$ which already got included as an endpoint. So we look for critical points where $A'(s) = 0$, i.e. $2s - \frac{2000}{s^2} = 0$, so we solve for s in $s = \frac{1000}{s^2}$, $s^3 = 1000$, $s = 10$.

At this stage we believe the answer is $s = 10$ and hence $h = \frac{500}{s^2} = 5$. But does that mimimize or maximize the area? Mathematically we could apply the first derivative test and see whether $A'(s)$ changes from negative to positive at $s = 10$, or more simply calculate $A''(10)$ and use the second derivative test. $A''(s) = 2 + \frac{4000}{s^3}$, so $A''(10) = 2 + 4 = 6$ which is positive. Hence $s = 10$ does correspond to a local minimum.

Alternatively we could reason as follows: As $s \to 0^+$, the area grows arbitrarily large, and as $s \to \infty$, the area also grows arbitrarily large. Since the area at intermediate points is finite, e.g. the area $A(10) = 300$

for our chosen dimensions, and $A(s)$ is continuous, there must be some s value minimizing it, and it must occur at a critical point, so that s value must be $s = 10$ and we do have a minimum.

Problem 8 (12 points)
Solve the Initial Value Problem

$$f''(x) = 6x - 4 \text{ and } f'(1) = -1 \text{ and } f(0) = 2$$

(a) First find f': Use the fact that $f'' = 6x - 4$ to find an infinite collection of possible functions, then use $f'(1) = -1$ to select the right one.

Answer: Since $f''(x) = 6x-4$, f' is a function whose derivative is $6x-4$, i.e. one of the infinitely many functions $\int (6x-4)dx = 3x^2-4x+C$ that are the antiderivatives of $6x-4$. Hence $f'(x) = 3x^2-4x+c$ for some number c. Then $f'(1) = 3 \times 1 - 4 \times 1 + c = c - 1$, but we have to have $f'(1) = -1$, so $c - 1 = -1$, i.e. $c = 0$. So $f'(x) = 3x^2 - 4x$.

(b) Now find f: Find a collection of functions that have for their derivatives whatever you got in (a), then use $f(0) = 2$ to select the right one.

Answer: Now we want a function f whose derivative is $3x^2 - 4x$, i.e. f is among the infinitely many functions $\int (3x^2 - 4x)dx = x^3 - 2x^2 + C$. So $f(x) = x^3 - 2x^2 + c$ for some number c. But we must have $f(0) = 2$: $f(0) = 0 - 0 + c = c$, so $c = 2$ is the only choice. Hence $f(x) = x^3 - 2x^2 + 2$.

(At this point we can easily check our work: $f'(x) = 3x^2 - 4x$ for this $f(x)$ so $f''(x) = 6x - 4$, as required. And $f(0) = 2$, and $f'(1) = 3 - 4 = -1$, so it meets the prescribed initial conditions.)