Answers to First Midterm Exam October 11, 2007

Problem 1:

(a) An object moves along the $y-$axis so that its position at time t (in seconds) is given by $y = t^3 - 2t + 1$ (in feet).

(i) What is its displacement (net distance moved) from $t = 0$ to $t = 2$?

Answer: At $t = 0$, $y = 0^3 - 2 \times 0 + 1 = 1$, and at $t = 2$, $y = 5$. So the displacement is $5 - 1 = 4$ feet.

(ii) What is its average velocity from $t = 0$ to $t = 2$? (include units)

Answer: Since it moved from 1 to 5 as t went from 0 to 2, the average velocity was $\frac{5-1}{2-0} = \frac{4}{2} = 2$ feet per second.

(iii) What is its instantaneous velocity at $t = 2$? (include units)

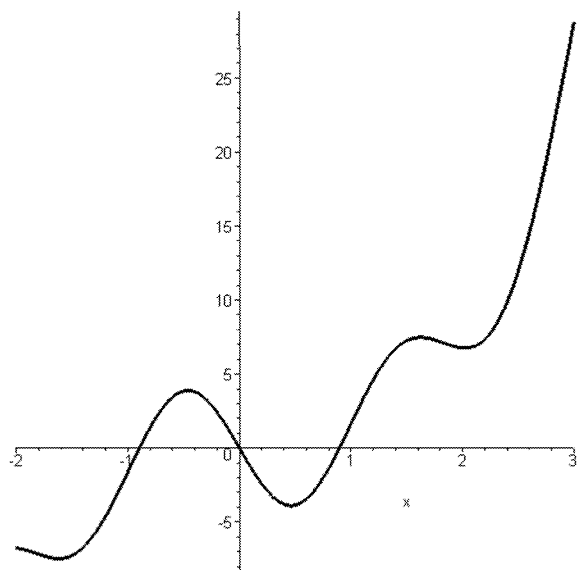
Answer: The derivative $y'(t) = 3t^2 - 2$ so at $t = 2$ we have $y' = 3 \times 2^2 - 2 = 10$ feet per second.

(b) Find the first, second, and third derivatives of $y = \sin(2x)$. (Be sure to indicate which is which!)

Answer: We have to remember the chain rule, each derivative will multiply on another 2 as the derivative of $2x$. So the derivatives, in order, are $2\cos(2x)$, $-4\sin(2x)$, and $-8\cos(2x)$.

Problem 2:

For the function $y = f(x)$ graphed to the right, answer (a)-(e) as true (T) or false (F), and supply a numeric answer for (f) and (g).



(a) There is a number c between 1 and 2 such that $f'(c) = 0$.

Answer: True: The graph goes up, then turns around and goes down, and where the tangent line is horizontal there will be a point where $f'(c) = 0$.

(b) There is a number c between 1 and 2 such that $f(c) = 0$.

Answer: False: The graph is above the x-axis throughout the interval $1 < x < 2$, so the function's value is never 0.

(c) If $2\frac{1}{4} < x < 3$, $f'(x) \geq 0$.

Answer: True: The function is increasing everywhere beyond just a little more than 2, so the derivative will be positive, actually greater than 0, and hence at least 0.

(d) The average rate of change of $f(x)$ from $x = -1$ to $x = 1$ is greater than 2.

Answer: False: The heights at the beginning ($x = -1$) and end ($x = 1$) of that interval are almost the same. The height at the right-hand end is very slightly positive, and the height at the left is very slightly negative, so the average rate of change is positive. But if it were greater than 2, the increase over that interval of length 2 would have to be at least $2 \times 2 = 4$ and clearly it is not.

(e) As x goes from -1 to 0, the derivative $f'(x)$ is decreasing consistently.

Answer: True: At the left end the slope is rather steeply positive. As we move from left to right through $[-1, 0]$ the slope decreases, going negative somewhere around $x = -\frac{1}{2}$.

(f) At approximately what value of x (within this picture) does the derivative $f'(x)$ take its largest value?

Answer: The graph seems most steeply sloped upward at the right hand side of the picture, where $x = 3$.

(g) At approximately what value of x (within this picture) does the derivative $f'(x)$ take its smallest (most negative) value?

Answer: The graph appears to be sloping downward most steeply at the origin, where $x = 0$.

Problem 3:

(a) Evaluate: $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$.

Justify each statement or calculation. You do not need to use an $\epsilon - \delta$ argument, but you should refer to things like "the limit of the sum of two functions...", basing your argument on theorems and limit rules proved in lecture and in the textbook.

Answer: If the function as given were continuous at $x = 3$ we could simply "plug in", but at $x = 3$ the denominator is zero so the rule does not even give a value for the function there. We rearrange algebraically: $\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3}$. In taking the limit as $x \to 3$, we don't care what happens at $x = 3$, and for any other value, where $x - 3 \neq 0$, we can divide $x - 3$ out of numerator and denominator. So while it would not be true to say $\frac{x^2-9}{x-3} = x + 3$, they are equal wherever $x \neq 3$ and hence $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3}(x + 3)$. That latter is the limit of a polynomial, which is continuous everywhere, so we can evaluate it at $x = 3$ and get $3 + 3 = 6$ as the desired limit.

(b) It is true that $\lim_{x \to 4}(-2x + 9) = 1$.

Use the $\epsilon - \delta$ definition of the limit to confirm this.

Answer: Since the graph of $y = -2x + 9$ is a straight line with slope -2, any horizontal change (i.e. a change in x) corresponds to a vertical change that is twice as large. So to keep the values of the function within ϵ of 1 we will want to keep the values of x within $\frac{\epsilon}{2}$ of 4. So for any $\epsilon > 0$ I choose $\delta = \frac{\epsilon}{2}$ which will also be greater than 0.

Then: If $0 < |x - 4| < \delta = \frac{\epsilon}{2}$, so $-\frac{\epsilon}{2} < x - 4 < \frac{\epsilon}{2}$, we multiply by 2 and get $-\epsilon < 2x - 8 < \epsilon$. But if $2x - 8$ is within that interval, so is $-2x + 8$, the negative of $2x - 8$, since the interval is symmetric about 0. So now we know $-\epsilon < -2x + 8 < \epsilon$. Adding 1 to each piece of that inequality, we have $1-\epsilon < -2x+9 < 1+\epsilon$, i.e. $-2x+9$ is within ϵ of 1. That is exactly what we need!

Problem 4:
Find the derivatives of:

(a) $$y = \frac{\sin(x)}{x^2}$$

Answer: You could either use the quotient rule or change the expression to $y = \sin(x)(x^{-2})$ and use the product rule. The quotient rule seems more natural.

$$y' = \frac{x^2\cos(x) - 2x * \sin(x)}{x^4} = \frac{x\cos(x) - 2\sin(x)}{x^3},$$

or various other forms it could be changed into.

(b) $$y = x^3\cos(3x-2)$$

Answer: We use the product rule and also notice we need the chain rule. $y' = 3x^2\cos(3x-2) + x^3[-3\sin(3x-2) = 3x^2\cos(3x-2) - 3x^3\sin(3x-2)$.

(c) $$y = (\sin(3x))^{100}$$

Answer: Using the power we might think $y' = 100(\sin(3x))^{99}$ but the chain rule says we must multiply by the derivative of $\sin(3x)$ and then the derivative of $3x$. So the answer is $100(\sin(3x))^{99} \times \cos(3x) \times 3 = 300\cos(3x)(\sin(3x))^{99}$.

(d) Find the third derivative of $y = \sin(2x)$.

(This is in the online version of the exam but was not in my file that supposedly produced that exam! I don't know what happened, so I don't know now whether it was on the exam that was printed out and given to the students!)

Answer: The first derivative, using the chain rule, will be $2\cos(2x)$. The second derivative will get another 2 multiplied on, $-4\sin(2x)$. The third derivative will be $-8\cos(2x)$. (As a partial check you could go to the fourth derivative, $16\sin(2x)$, and remember that taking derivatives of the sine or cosine you go in a cycle, repeating much of it every four steps.)

Problem 5:
Find an equation for the tangent line to the graph of

$$x\sin(2y) = y\cos(2x)$$

at the point $(\frac{\pi}{4}, \frac{\pi}{2})$.
Answer: For the slope we need $\frac{dy}{dx}$. It does not look practical to solve for y as a function of x, so we differentiate implicitly. Taking the derivatives of the terms in the equation we are given, and using the product and chain rules, we have $\sin(2y) + 2x\cos(2y)\frac{dy}{dx} = \frac{dy}{dx}\cos(2x) - 2y\sin(2x)$. Separating the terms that do have $\frac{dy}{dx}$ from the others we get $\frac{dy}{dx}(2x\cos(2y) - \cos(2x)) = -\sin(2y) - 2y\sin(2x)$. Dividing we have

$$\frac{dy}{dx} = -\frac{\sin(2y) + 2y\sin(2x)}{2x\cos(2y - \cos(2x)}.$$

When $x = \frac{\pi}{4}$ and $y = \frac{\pi}{2}$ we get

$$\frac{dy}{dx} = -\frac{\sin\pi + \pi\sin\frac{\pi}{2}}{\frac{\pi}{2}\cos\pi - \cos\frac{\pi}{2}} = -\frac{0+\pi}{-\frac{\pi}{2} - 0} = 2.$$

. So now we know we need the line through $(\frac{\pi}{4}, \frac{\pi}{2})$ with slope 2. The easiest way to write out that equation is $y - \frac{\pi}{2} = 2(x - \frac{\pi}{4})$ which can be rewritten as $y = 2x$.

Problem 6:
Again there is a difference between the posted exam and my files for what the exam was, as in problem 4. This time there are two completely different problems! The first one here is what I thought the problem was:

(a) Evaluate the limit. You must justify your answer, but you do not need to use precise arguments involving δ.

$$\lim_{x\to 2^-} \frac{3}{x-2}$$

Answer: The answer is $-\infty$: For any number N we can make x so close to 2, but less than 2, that the difference between x and 2 is less than $\frac{1}{|N|}$, and since $x < 2$ we have $x - 2$ negative, so $\left|\frac{3}{x-2}\right|$ is more than 3 times N and hence certainly greater than N. So we can make the absolute value of $\frac{3}{x-2}$ arbitrarily large as x gets close to 2 on the left, but the fraction itself is negative, so the limit is $-\infty$.

(b) Use the definition of the derivative as a limit to find the derivative of $3x^2 - 2x + 5$.

Answer: We evaluate $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$, where $f(x) = 3x^2 - 2x + 5$. Then $f(x+h) = 3(x+h)^2 - 2(x+h) + 5 = 3(x^2 + 2xh + h^2) - 2(x+h) + 5 = 3x^2 + 6xh + 3h^2 - 2x - 2h + 5$. Putting these together we need

$$\lim_{h\to 0} \frac{(3x^2 + 6xh + 3h^2 - 2x - 2h + 5) - (3x^2 - 2x + 5)}{h} = \lim_{h\to 0} \frac{6xh + 3h^2 - 2h}{h}.$$

For any h other than 0 we can cancel an h from numerator and denominator, so the limit is the same as $\lim_{h\to 0}(6x + 3h - 2)$ which is $6x - 2$.

And here is the problem that was in the online version of the exam: I don't know what was printed and given to the students at the exam!

(a) Evaluate the limit. You must justify your answer, but you do not need to use precise arguments involving δ or ϵ.

$$\lim_{x\to 0} \frac{3x + \sin(x)}{4x}$$

Answer: Theorems we have say that (assuming the limits exist) we could rewrite the limit algebraically as $\lim_{x\to 0}\left(\frac{3x}{4x}\right) + \frac{1}{4}\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)$. For any $x \neq 0$, the first fraction is $\frac{3}{4}$, so the first limit is $\frac{3}{4}$. The second limit we proved was 1, so the complete result is $\frac{3}{4} + \frac{1}{4} \times 1 = 1$.

(b) Use the definition of the derivative as a limit to find the derivative of $3x^2 - 2x + 5$.

Answer: We evaluate the limit $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$, for $f(x) = 3x^2 - 2x + 5$. Then $f(x+h) = 3(x+h)^2 - 2(x+h) + 5 = 3x^2 + 6xh + 3h^2 - 2x - 2h + 5$, so the numerator of that fraction is $(3x^2 + 6xh + 3h^2 - 2x - 2h + 5) - (2x^2 - 2x + 5) = 6xh + 3h^2 - 2h$. Thus the limit becomes $\lim_{h\to 0} \frac{h(6x + 3h - 2)}{h}$. For any $h \neq 0$, the only values we need to consider in that limit, we can divide h out of both numerator and denominator (as indicated by the factoring) and cancel. So that limit is the same as $\lim_{h\to 0}(6x+3h-2)$. We could be formal about evaluating that limit or we can just say that the $6x$ and -2 terms don't change with h while the term $3h$ goes to 0, so the limit (and hence the derivative) is $6x-2$.

Problem 7:

(a) The radius of a sphere is growing at a rate of 3 inches per second.

How fast is the volume of the sphere changing at the instant when the radius is 10 inches?

Answer: Differentiating the relation $V(t) = \frac{4}{3}\pi(r(t))^3$, we have $V' = \frac{4}{3}\pi(3r^2) = 4\pi r^2 r'$. When $r = 10$ inches, and $r' = +3$ inches/second, $V' = 4\pi(100)(3) = 1200\pi$ cubic inches/second.

(b) The function defined by $f(x) = \frac{3x^2-7x-20}{x-4}$ is not continuous at $x = 4$. Construct a new function $g(x)$ which gives the same value for any $x \neq 4$ but such that $g(x)$ is continuous at $x = 4$.

Answer: Evaluating the given $f(x)$ at $x = 4$ would require dividing by $4-4 = 0$, so the function is not defined at that point. But if it has a limit at that point we could make $g(x)$ be the same as $f(x)$ everywhere else and have $g(x)$ take that limit as its value at $x = 4$. Evaluating

$$\lim_{x\to 4} f(x) = \lim_{x\to 4} \frac{3x^2-7x-20}{x-4} = \lim_{x\to 4} \frac{(3x+5)(x-4)}{x-4} = \lim_{x\to 4}(3x+5) = 17.$$

(Using (a) the fact that we don't need to include $x - 4 = 0$ so we can divide that out, and (b) $3x + 5$ is a polynomial and hence continuous everywhere.) So now we can define $g(x)$ in either of two equivalent ways. The way we started out toward is: $g(x) = f(x)$ if $x \neq 4$, and $g(4) = 17$. Or, as a by product of the way we evaluated the limit, we see it has the same effect just to let $g(x) = 3x + 5$.

Problem 8:

(a) Find the linearization of $f(x) = \sin(x)$ at $x = \pi$.

Answer: $f'(x) = \cos(x)$ takes the value -1 at $x = \pi$. $f(x)$ itself takes the value $\sin(\pi) = 0$. So the linearization is $L(x) = (-1)(x-\pi) + 0 = -x + \pi$.

(b) Use that linearization to estimate $\sin(\pi + 0.1)$.

Answer: We evaluate L at $\pi+0.1$, getting $-(\pi+0.1)+\pi = -0.1$. (The actual value of $\sin(\pi+0.1)$ is about $-0.09983353\ldots$, an error of only about 0.00017.)