Your Name: ____

Circle your TA's name:

Ben Akers **Adam Berliner** **Ben Ellison**

James Hunter **Peter Spaeth**

Mathematics 221, Fall 2003 **Lecture 2 (Wilson)**

Final Exam December 16, 2003

Write your answers to the twelve problems in the spaces provided. If you must continue an answer somewhere other than immediately after the problem statement, be sure (a) to tell where to look for the answer, and (b) to label the answer wherever it winds up. In any case, be sure to make clear what is your final answer to each problem.

Wherever applicable, leave your answers in exact forms (using $\frac{\pi}{3}$, $\sqrt{3}$, $\cos(0.6)$, and similar numbers) rather than using decimal approximations. If you use a calculator to evaluate your answer be sure to show what you were evaluating!

There is a problem on the back of this sheet: Be sure not to skip over it by accident!

There is scratch paper at the end of this exam. If you need more scratch paper, please ask for it.

You may refer to notes you have brought in on up to three index cards, as announced at the class website.

BE SURE TO SHOW YOUR WORK, AND EXPLAIN WHAT YOU DID. YOU MAY RECEIVE REDUCED OR ZERO CREDIT FOR UNSUBSTANTIATED ANSWERS. ("I did it on my calculator" and "I used a formula from the book" (without more details) are not sufficient substantiation...)

Problem	Points	Score
1	17	
2	17	
3	16	
4	20	
5	14	
6	14	

Problem	Points	Score
7	18	
8	21	
9	15	
10	16	
11	16	
12	16	
TOTAL	200	

Problem 1 (17 points)
Let $f(x) = 3x^2 - 4x + 4$, $a = -1$, and $b = 4$. (These are used throughout the rest of this problem.)

(a) The mean value theorem for derivatives tells us that there is some number c in the interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
Find a number c that does what that theorem guarantees.

(b) The mean value theorem for integrals tells us that there is some number d in the interval (a, b) such that $f(d)$ is the same as the average value of the function f on the interval $[a, b]$.

(i) Find the average value of f on $[a, b]$.

(ii) Find a number d that does what this theorem guarantees.

Problem 2 (17 points)
Find the general solution to the differential equation

$$x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}.$$

(Hint 1: First divide through the equation by x.
Hint 2: $e^{3\ln x} = (e^{\ln x})^3$.)

Problem 3 (16 points)

Let $f(x) = 3x - 2$. Use the $\epsilon - \delta$ definition of limit to show $\lim_{x \to 4} f(x) = 10$.

Note: Just "plugging in" 4, or any other argument which assumes $f(x)$ is continuous, will get no credit. Also, remember that the definition requires more than just finding δ: You should show that the choice you make works.

Problem 4 (20 points)

Take the indicated derivatives:

(a) $\dfrac{dy}{dx}$, for $y = \ln(\sin(x) + 2)$.

(b) $\dfrac{d^2 f}{dx^2}$, for $f(x) = 3e^{2x} + 5x^3$.

(c) $D_x f$, for $f(x) = \dfrac{1}{x} \arcsin(x^2 - 2)$.

(d) $f'(x)$, for $f(x) = \displaystyle\int_2^{x^2-1} \tan^{-1}(t)\, dt$.

Problem 5 (14 points)

Let $f(x) = e^x$.

(a) Find an equation for the tangent line to the graph of $f(x)$ at $x = 1$.

(b) Use a tangent line approximation (also known as a linear approximation) to estimate $e^{\frac{1}{2}}$.

(c) By thinking about the shape of the graph of e^x, tell whether you think your estimate in (b) will be higher or lower than the actual value of $e^{\frac{1}{2}}$. Explain your reasoning.

Problem 6 (14 points)
Let $f(x) = x^2 - 2$.

(a) Set up a Riemann sum approximating $\int_1^5 f(x)\,dx$ which uses a partition of $[1,5]$ into 4 equal subintervals and evaluation of f at the right end of each subinterval. Show the numbers to be added up explicitly, e.g. $\frac{5}{4}$, without variables in them, i.e. not as $\frac{1}{2}x_i$ or anything like that. You should add them up, thus arriving at an approximate value for the integral.

(b) Evaluate $\int_1^5 f(x)\,dx$ using the second Fundamental Theorem of Calculus.

(c) The answer you got in (b) is either less than or more than the answer you got in (a). Explain why the difference was in the direction it was (e.g., if (b) came out larger, why that should be the case), making reference to the direction the graph of f is curving and the choice of where to evaluate $f(x)$ in part (a).

Problem 7 (18 points)
Let $f(x) = 3x^4 - 20x^3 + 36x^2 + 7$.

(a) Find all critical points of $f(x)$ on the interval $[-1, 4]$.

(b) For each critical point in the open interval $(-1, 4)$, tell if it is a local maximum, a local minimum, or neither. Give calculus-based reasons for your answers, not just what you see on a calculator-produced graph.

(c) Find all points of inflection of the graph of $f(x)$. (Remember that our text defines a point of inflection to be the point on the graph, i.e. it has two coordinates!)

Problem 8 (21 points)
Evaluate the integrals:

(a) $$\int \frac{\sin(x)}{1+\cos(x)}\,dx$$

(b) $$\int_0^{\ln(2)} x\,e^{3x^2}\,dx$$

(c) $$\int \cos(5x)\,dx$$

Problem 9 (15 points)
Find the area of the region in the plane which is bounded by (i) $y = e^{\frac{x}{2}}$, (ii) $y = 2x$, (iii) $x = 1$, and (iv) $x = 4$.

Problem 10 (16 points)

A Petri dish contains a colony of bacteria. If $f(t)$ gives the amount of the colony at time t, the rate of growth of f is proportional to the value of f itself.

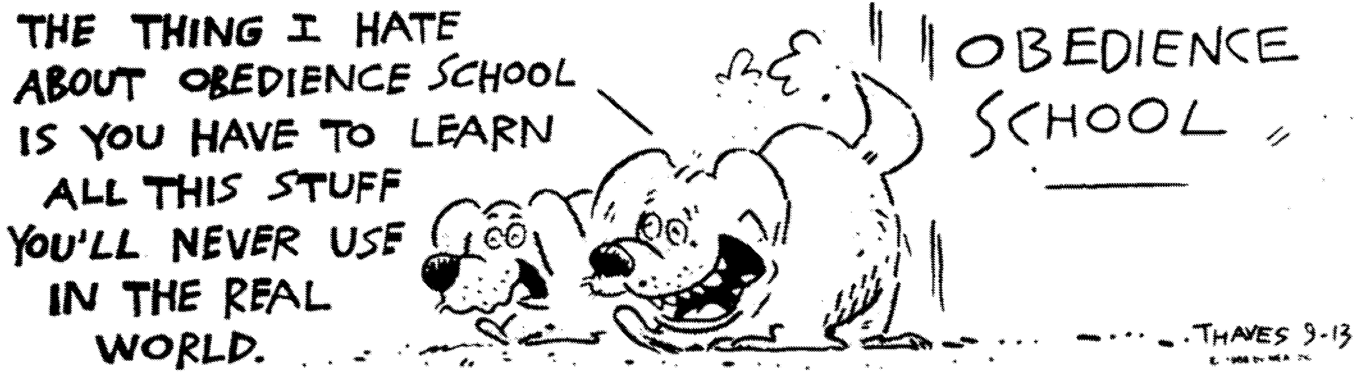
An experimenter observes that there are 0.2 grams of bacteria five hours into an experiment, and 0.8 grams after the experiment has been going for 15 hours.

(a) How many grams were there at the start of the experiment?

(b) How many grams $f(t)$ will there be for an arbitrary time t, assuming the same conditions continue to hold? (Your answer should be a function involving t, not the value of f at some specific time t you have chosen!)

Problem 11 (16 points)
An object is being moved against a resisting force. It starts at a certain position, and for the first three feet it is moved it encounters a fixed resisting force of 25 pounds. After that the force starts changing. For the next three feet the force is $22 + x$ pounds, where x is the distance the object has moved from its starting position.
When the object has been moved six feet (the first three with the constant force and the last three with the varying force) how much work has been done?

Problem 12 (16 points)
Use an integral to find the volume of the solid generated by rotating about the y-axis the region bounded by $y = x$, $x = 0$, and $y = 2$.
(You may recognize this shape and know a formula for its volume, but that should only be used as a check on your answer: You must show how to find the volume using an integral.)



SCRATCH PAPER