Problem 1

Let $f(x) = x^3 - 6x^2 - 15x + 1$. Find all local and global maxima and minima of $f(x)$ on the interval $[-3, 6]$.

Answer: We look for critical points: Since f is given by a polynomial, it is differentialble everywhere. Hence there are no singular points. There are the two endpoints $x = -3$ and $x = 6$. This leaves the stationary points to be found. $f'(x) = 3x^2 - 12x - 15 = 3(x^2 - 4x - 5) = 3(x-5)(x+1)$, so the values making $f'(x) = 0$ are $x = 5$ and $x = -1$. Both of those are in the interval $[-3, 6]$. So, in order, the critical points are -3, -1, 5, and 6.

At the two stationary points we can use either the first derivative test (Theorem A on page 175) or the second derivative test (Theorem B on page 176). I will use the second derivative test. $f''(x) = 6x - 12$. At the first stationary point $x = -1$, we have $f''(-1) = -18 < 0$. Hence there is a local maximum at $x = -1$, where $f(-1) = 9$. At the other stationary point $x = 5$ we have $f''(5) = 18 > 0$ so there is a local minimum there, where $f(5) = -99$.

Now at the two endpoints we consider the slope. At $x = -3$ we have $f'(-3) = 48 > 0$ so the graph is sloping upward at the left end point, which must be a local minimum. At this point we have $f(-3) = -35$. Lastly at the right end point we have $f'(6) = 21 > 0$, so this must be a local maximum. (Note the same sign for the derivative gives different answers at the left and right ends!) At this point we have $f(6) = -89$.

Lastly, for global max/min on this interval, we compare the values of f at the four candidate points. The largest is $f(-1) = 9$, and the smallest is $f(5) = -99$. Hence the global maximum occurs at $x = -1$ and the global minimum at $x = 5$.

Problem 2

Let $f(x) = x + 2$.

(a) Set up and evaluate the Riemann sum for $f(x)$ which results from partitioning the interval $[-1, 1]$ into 4 equal-width subintervals and using the left end of each subinterval as the sample point.

Answer: If we divide the interval $[-1, 1]$ into four equal parts, they will be $[-1, -\frac{1}{2}]$, $[-\frac{1}{2}, 0]$, $[0, \frac{1}{2}]$, and $[\frac{1}{2}, 1]$, each having width $\Delta x = \frac{1}{2}$. Using the left end of each as the sample point, our sum is $f(-1)\Delta x + f(-\frac{1}{2})\Delta x + f(0)\Delta x + f(\frac{1}{2})\Delta x = (1 + \frac{3}{2} + 2 + \frac{5}{2})\Delta x = 7\Delta x = \frac{7}{2}$.

(b) Set up and evaluate the Riemann sum for $f(x)$ which results from partitioning the interval $[-1, 1]$ into n equal-width subintervals (not for a specific value of n), and using the left end of each subinterval as the sample point. (Your answer when evaluating this sum should be a formula that involves n and possibly some summation signs Σ.)

Answer: The n subintervals will each have width $\Delta x = \frac{2}{n}$, since the whole interval is two units long. They will be of the form $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$, where $x_0 = -1$ and $x_n = 1$: Counting off from -1 in steps of Δx we see that $x_i = -1 + i\Delta x = -1 + \frac{2i}{n}$. The entry in the sum corresponding to a subinterval will thus be $f(x_{i-1})\Delta x = f(-1 + \frac{2(i-1)}{n})(\frac{2}{n})$ $= (-1 + \frac{2i-2}{n} + 2)(\frac{2}{n}) = \frac{2}{n} - \frac{4}{n^2} + \frac{4}{n^2}i$. Collecting these in the sum gives $\sum_{i=1}^{n}\left(\frac{2}{n} - \frac{4}{n^2} + \frac{4}{n^2}i\right)$.

Separating the parts of this sum that do have the variable i in them from those that don't, and factoring out constants (n is a constant for any particular version of this sum, although we will let it change in the next part of the problem), we get $\left(\frac{2}{n} - \frac{4}{n^2}\right)\sum_{i=1}^{n} 1 + \frac{4}{n^2}\sum_{i=1}^{n} i$. It would be an acceptable answer for (b) in this form, or the previous one, but for (c) we are going to need the sums evaluated so we go ahead and do that now: The first of those summations yields n and the second $\frac{n(n+1)}{2}$. Thus our sum amounts to $n\left(\frac{2}{n} - \frac{4}{n^2}\right) + \frac{n^2+n}{2}\frac{4}{n^2} = 2 - \frac{4}{n} + 2 + \frac{2}{n} = 4 - \frac{2}{n}$.

$\int_{-1}^{1}(x+2)\,dx$, and you can check your answer that way, but you must show how to get it from the limit of Riemann sums in order to receive credit for this problem.

Answer: We only need to take the limit as $n \to \infty$ of the result $4 - \frac{2}{n}$ that we got in (b): This is clearly 4.

Problem 3

Solve the initial value problem

$$\frac{dy}{dx} = \frac{x^2}{y}, \qquad \text{for } y > 0, \qquad \text{with} \qquad y(0) = 3.$$

Show explicitly the general solution to the differential equation and then how you pick the particular solution meeting the initial condition.

Answer: First we rearrange the equation as $y\,dy = x^2\,dx$. Now integrate both sides and get $\frac{1}{2}y^2 = \frac{1}{3}x^3 + C$. This is one form of the general solution: Writing it more explicitly as a function $y(x)$ we have $y = \pm\sqrt{\frac{2}{3}x^3 + C}$.

Continuing with the second form and using the initial condition $y(0) = 3$, a positive number, we see that we must use the positive square root, so $y = \sqrt{\frac{2}{3}x^3 + C}$. Substituting that initial condition gives $3 = \sqrt{0 + C}$ or $C = 9$. Hence the solution to the equation which also fits the initial condition is $y = \sqrt{\frac{2}{3}x^3 + 9}$.

Problem 4

Use a linear approximation to $f(x) = \sqrt[3]{x}$ to approximate $\sqrt[3]{7}$.

Answer: We construct the linear approximation $L(x)$ based at $x = 8$. The derivative $f'(x) = \frac{1}{3}x^{-2/3}$: At 8 we get $f'(8) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$. The function value $f(8) = 2$. Hence $L(x) = 2 + \frac{1}{12}(x - 8)$.
Evaluating $L(7)$ we get $2 + \frac{1}{12}(7 - 8) = 2 - \frac{1}{12} = \frac{23}{12}$.
As a check: $\frac{23}{12} \approx 1.91666$, while a calculator gives $\sqrt[3]{7} \approx 1.91293$. Not bad for such an easy calculation!

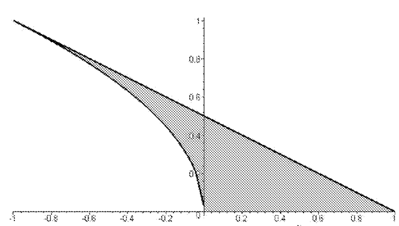
Problem 5

(a) Evaluate $\displaystyle\int_{-\frac{\pi}{4}}^{0} \sin^2(2x)\cos(2x)\,dx$.

Answer: This cries out for a substitution. Let $u = \sin(2x)$. Then $du = 2\cos(2x)dx$. When $x = -\frac{\pi}{4}$, $u = \sin(-\frac{\pi}{2}) = -1$. When $x = 0$, $u = 0$. Thus the integral becomes

$$\int_{-1}^{0} u^2 \frac{1}{2}\,du = \frac{1}{6}u^3\Big]_{-1}^{0} = 0 - \left(\frac{1}{6}(-1)^3\right) = \frac{1}{6}.$$

(b) Find the area of the region shown, bounded by parts of the curve $x = -y^2$, the line $2y = 1-x$, and the x-axis.



conditions change at $x=0$: The lower edge changes from being the curve to being the x-axis. So instead we set this up as an integral from horizontal slices. The lowest y value in this region is $y=0$ and the highest, at the upper left corner, is $y=1$. At any height y between those extremes, a horizontal slice through the region extends from $x=-y^2$ on the left to $x=2y-1$ on the right, so a slice that is Δy tall has area $(2y-1-(-y^2))\,\Delta y$. Summing and taking the limit we get the area as

$$\int_0^1 \left(y^2+2y-1\right)\,dy = \left[\frac{y^3}{3}+y^2-y\right]_0^1 = \frac{1}{3}+1-1=\frac{1}{3}.$$

It is possible to do this with vertical slices, but you would have to set up two integrals and add the results. You should get the same answer.

Problem 6

Let $f(x)=x^2-6x+7$.

(a) Find the average (mean) value of $f(x)$ on the interval $[2,5]$.

Answer:

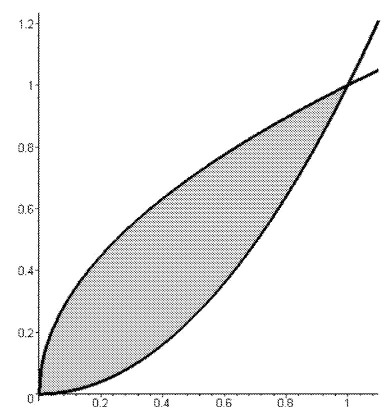
$$\text{The average is } \frac{1}{(5-2)}\int_2^5 (x^2-6x+7)\,dx = \frac{1}{3}\left[\frac{x^3}{3}-3x^2+7x\right]_2^5 = \frac{1}{3}\left[\frac{5}{3}-\frac{14}{3}\right] = -1.$$

(b) The Mean Value Theorem for Integrals guarantees the existence of a number c within the interval $[2,5]$ (i.e. c is not either of the endpoints) such that $f(c)$ is the average value. Find such a number c.

Answer: We solve $f(c)=-1$, i.e. $c^2-6c+7=-1$. The roots of that equation are 2 and 4. Since we are asked for a value c within the interval $[2,5]$ we should not use 2. Hence the answer is $c=4$.

Problem 7

The region between the graphs of $y=\sqrt{x}$ and $y=x^2$ is shown at the right. Set up and evaluate an integral to compute the volume of the solid that results when this region is rotated about the x-axis.



Answer: The region extends in the x direction from $x=0$ to $x=1$. Since we are rotating it about the x-axis, the radius of the resulting solid at any given x value will be determined by y. A slice through the solid at x results in a "washer", a circular disk with a circular hole in its center: The outer radius of the washer is determined by the y value on the upper curve, $y=\sqrt{x}$, while the radius of the hole is determined by the y value on the lower curve, $y=x^2$. The area of the washer is the area of a circle of radius $\sqrt{x}$ less the area of a circle of radius x^2, i.e. $\pi(\sqrt{x})^2-\pi(x^2)^2$. Simplifying we get the area of the slice $A(x)=\pi(x-x^4)$. Now we integrate to get the volume as

$$\int_0^1 \pi(x-x^4) = \pi\left[\frac{x^2}{2}-\frac{x^5}{5}\right]_0^1 = \frac{3\pi}{10}.$$

Suppose $f(x)$ is a function with the following properties:

- $f(0) = 2$
- $f'(x) < 0$ for $-3 \le x < 0$
- $f'(0) = 0$
- $f'(x) < 0$ for $0 \le x < 6$
- $f'(x) > 0$ for $6 < x \le 10$
- $f''(x) > 0$ for $-3 \le x < 0$
- $f''(0) = 0$
- $f''(x) < 0$ for $0 < x < 4$
- $f''(4) = 0$
- $f''(x) > 0$ for $4 < x \le 10$

Draw on the axes below the graph of a function with these properties.
Answer: We are only given information for $-3 \le x \le 10$. The graph could do anything outside that interval, but within it we know a lot: We know one point on the graph, $(0, 2)$, from the first property given. We also know the graph goes through that point horizontally since $f'(0) = 0$. But $f'(x)$ is negative on both sides of zero, so the graph must be going downhill, level off briefly, then head down again. The first derivative does not become positive until x gets to 6, where the graph will start climbing. Now from the information on the second derivative we see the graph is concave upward for $-3 \le x < 0$, has a point of inflection when $x = 0$, is concave downward from there to $x = 4$ where it has another inflection point, and finally is concave upward from there to the right.
There are many possible correct answers, but they all must fit this description. Here is one version: