PROBLEM 1:

(a) Find the derivative $f'(x)$ for $f(x) = \frac{x^3+2x+1}{x^2-3}$.

ANSWER: We use the quotient rule. The derivative of the numerator is $3x^2+2$ and the derivative of the denominator is $2x$, so we get

$$\frac{(3x^2+2)(x^2-3)-(2x)(x^3+2x+1)}{(x^2-3)^2} = \frac{x^4-11x^2-6-2x}{(x^2-3)^2}.$$

(b) Find the derivative $f'(x)$ for $f(x) = \sqrt{x^3-2x+1}$.

ANSWER: Here we have the square root function applied to the results from evaluating the polynomial inside, so we need to remember to use the chain rule. We think of the square root as the $\frac{1}{2}$ power, and we get

$$\frac{1}{2}(x^3-2x+1)^{-\frac{1}{2}}(3x^2-2) = \frac{3x^2-2}{2\sqrt{x^3-2x+1}}.$$

(c) Find the second derivative $f''(x)$ for $f(x) = \cos(x^2+1)$.

ANSWER: We take the first derivative, using the chain rule, and get $f'(x) = -2x\sin(x^2+1)$. Now we take the derivative of that, using the product and chain rules, and get

$$f''(x) = -2\sin(x^2+1) - 2x\cos(x^2+1)(2x) = -2\sin(x^2+1) - 4x^2\cos(x^2+1).$$

PROBLEM 2:
Use the definition of the derivative as a limit to find $f'(x)$, for $f(x) = 2x^2+x$.
ANSWER: There are several different forms in which you can write this limit. I will use

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, so for this function we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{[2(x+h)^2+(x+h)]-[2x^2+x]}{h} \\ &= \lim_{h\to 0} \frac{2x^2+4xh+2h^2+x+h-2x^2-x}{h} \\ &= \lim_{h\to 0} \frac{4xh+2h^2+h}{h}. \end{aligned}$$

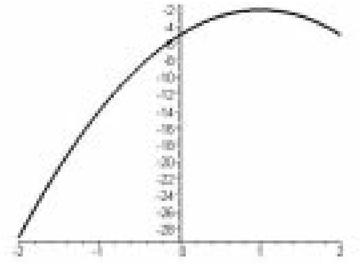
In taking this limit we do not need to consider $h=0$, just h "near" 0, so we can divide h out of the numerator and cancel and get $f'(x) = \lim_{h\to 0}(4x+2h+1)$. So far as the limit variable h is concerned, $4x$ and 1 are constants, and $\lim_{h\to 0} h = 0$, so we get $f'(x) = 4x+1$.

PROBLEM 3:
Road N goes north and south, and road E goes east and west. A car is travelling north on N at 50 miles per hour. Another car is travelling east on E at 60 miles per hour. How fast is the distance between them changing at the instant when the first car is 20 miles south of where the roads meet and the second car is 15 miles east of where the roads meet? Is the distance between the cars increasing or decreasing at that moment?

ANSWER: Let $y(t)$ be the distance the first car is north of the intersection and $x(t)$ the distance the second car is east of the intersection at time t. If we measure the distances all in miles and time in hours, the data given tell us $y'(t) = 50$ miles per hour and $x'(t) = 60$ miles per hour. If we let $s(t)$ be the distance between the cars at time t, by the Pythagorean theorem we have $s^2 = x^2 + y^2$. Differentiating we get the relation $2s(t)\, s'(t) = 2x(t)\, x'(t) + 2y(t)\, y'(t)$ which can be simplified by dividing every term by 2. What we need to find is s' at the instant when $x = 15$ and $y = -20$. (Note the sign! It would also have worked to let y be distance south of the intersection, in which case y at this instant would be $+20$, but in that case y' would have been -50 since y was decreasing. You could go either way but you must be consistent.) At the moment when $x = 15$ and $y = 20$, $s = \sqrt{15^2 + 20^2} = \sqrt{625} = 25$. Now putting these facts into the relation we derived, $25\, s' = 15 \times 60 + (-20) \times 50 = 900 - 1000 = -100$, so $s' = -4$ miles per hour.
Now we can see the answers: The distance between the cars is decreasing 4 miles per hour.

PROBLEM 4:
Of the eight functions graphed below, four are the derivatives of the other four. All are plotted with the same horizontal scale, but the vertical scales vary.

Pair the functions with their derivatives.

A


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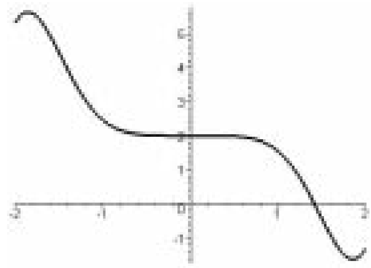

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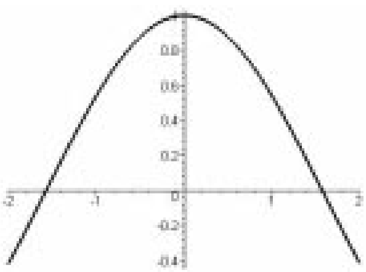

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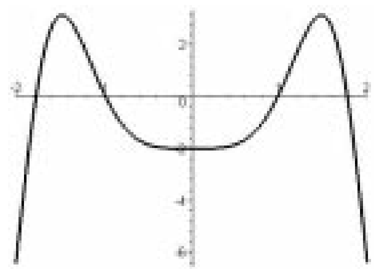

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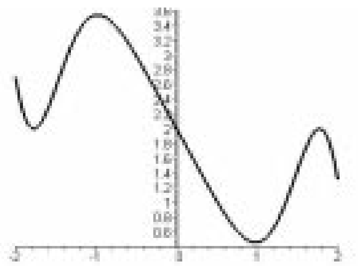

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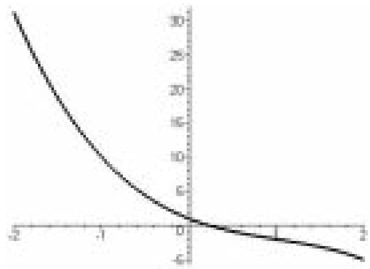

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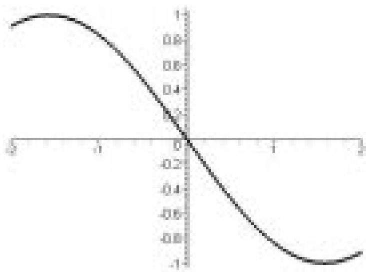
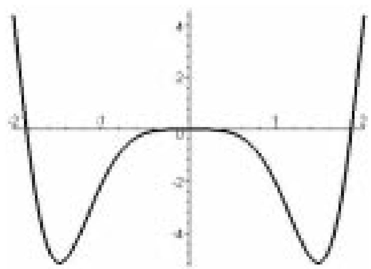


H


ANSWER: You could write the answers in a different order, but they will amount to the following:
Function A is the derivative of function F.
Function C is the derivative of function G.
Function D is the derivative of function E.
Function H is the derivative of function B.
Here are some sample arguments: You might have thought of this in several different ways.
Looking at graph A, note that it is everywhere negative (the highest point on the vertical scale is only -2). If that is the derivative for one of the others, that other will have to be decreasing everywhere: Is there such a graph? Yes, F is consistently decreasing. And F is close to horizontal just to the right of the y-axis, where A gets closest to zero. To either side of that region F slopes down more strongly and A is more negative. So we can say that A is the derivative of F.
If we look at C, it could either be a derivative or have one of the others for its derivative. If it were the derivative of another, that other would have to be horizontal at about $x = -1.6$ and $x = +1.6$, and D, E, and H all might seem to fit that, but each of those also is horizontal somewhere in between and C is not zero anywhere in between. So, instead, something else is presumably the derivative of C. Since C is increasing everywhere to the left of $x = 0$ and decreasing to the right, its derivative will be positive on the left and negative on the right. The only graph fitting that description is G, and so we decide G is the derivative of C. (In fact G is a graph of $-\sin(x)$ and C is a graph of $\cos(x)$.)

PROBLEM 5:

Let $f(x) = \dfrac{x^2 - 3x - 10}{x + 2}$.

(a) Then $f(x)$ is not continuous at $x = -2$. Why? Tell how it fails the definition of continuity at that point.

ANSWER: For f to be continuous at -2, f must be defined at -2 and this function is defined at all other numbers but not at -2.

(b) Now change the definition of $f(x)$ in such a way that your new version is continuous at $x = -2$. The new function should agree with the old one for most values of x.

ANSWER: For any x other than $x = -2$, f is defined and $f(x) = \frac{(x+2)(x-5)}{x+2} = x - 5$. Hence $\lim_{x\to -2} f(x) = -2 - 5 = -7$. If we change the definition of f to

$$f(x) = \begin{cases} \frac{x^2-3x-10}{x+2} & \text{for } x \neq -2 \\ -7 & \text{for } x = -2 \end{cases}$$

we have a new function which is the same as f at all points except $x = -2$ and which is continuous at $x = -2$.

PROBLEM 6:
Find the point (x_0, y_0) on the graph of $y = \frac{x}{x-3}$ (for $x > 3$) such that the tangent line to the graph at (x_0, y_0) passes through the point $(9, 1)$.
ANSWER: Note that (x_0, y_0) is on the graph and so $y_0 = x_0/(x_0 - 3)$.
Take the derivative

$$y' = \frac{(x-3)(1) - x(1)}{(x-3)^2} = \frac{-3}{(x-3)^2}$$

to use for the slope of the tangent line. Now we know the slope of the tangent line through (x_0, y_0) will be $-3/(x_0 - 3)^2$. Since we also know it goes through $(x_0, x_0/(x_0 - 3))$, the line must have equation

$$y - \frac{x_0}{x_0 - 3} = \frac{-3}{(x_0 - 3)^2}(x - x_0) \qquad \text{or} \qquad y(x_0 - 3)^2 - x_0(x_0 - 3) = -3(x - x_0).$$

But we also want the line to go through $(9, 1)$, and substituting $x = 9$ and $y = 1$ in that equation we get

$$1(x_0 - 3)^2 - x_0(x_0 - 3) = -3(9 - x_0) \qquad \text{or} \qquad x_0^2 - 6x_0 + 9 - x_0^2 + 3x_0 = -27 + 3x_0,$$

which simplifies to $x_0 = 6$. Then $y_0 = 2$ and the point is $(6, 2)$.

PROBLEM 7:
Use the definition of limit in terms of ϵ and δ to justify the statement

$$\lim_{x \to 4} (3x - 2) = 10.$$

You should both show what to use for δ and also demonstrate that your choice of δ does what is needed.
ANSWER: Given any $\epsilon > 0$, we need to show how to choose δ that guarantees that whenever $0 < |x - 4| < \delta$ we can be sure that $|(3x - 2) - 10| < \epsilon$. Now $|(3x - 2) - 10| = |3x - 12| = 3|x - 4|$, so if let δ be $\frac{1}{3}$ as big as ϵ that should work. This much was essentially scratch work, leading us to a good way to pick δ. So we use that choice and see what happens.
For any $\epsilon > 0$, let $\delta = \frac{\epsilon}{3}$. Then if $0 < |x - 4| < \delta = \frac{\epsilon}{3}$, $0 < 3|x - 4| < \epsilon$. Hence $|3x - 12| < \epsilon$, so this choice of δ does work in the definition, and we have justified the given limit.

PROBLEM 8:
A point on a rotating bicycle tire goes up and down, and its height at time t (in seconds), is given by $y = 26 \sin t$ (in inches). (We ignore its horizontal motion in this problem.)

(a) What is the average velocity of the point on the tire, between the times $t = 0$ and $t = \pi$?

ANSWER: The average velocity will be how far the point has moved divided by how much time went by. At $t = 0$ the position is $y = 26 \sin 0 = 0$, and at $t = \pi$ the position is $y = 26 \sin \pi = 0$, so the point is back to exactly where it started. Hence the average velocity is $\frac{0-0}{\pi} = 0$ inches per second.

(b) What is the instantaneous velocity of the point at the instant when $t = \frac{\pi}{4}$?

ANSWER: The instantaneous velocity will be the value of the derivative y' at that instant. Taking the derivative gives $y' = 26 \cos t$ in general, and when $t = \frac{\pi}{4}$ we have $y' = 26 \cos \frac{\pi}{4}$. That is an acceptable answer. You can also use $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and rewrite the answer as $13\sqrt{2}$ inches per second.